RATHINAM TECHNICAL CAMPUS

FINITE ELEMENT ANALYSIS

UNIT 1

BASICS OF FINITE ELEMENT METHOD

1.1 FUNDAMENTAL CONCEPTS OF ENGINEERING ANALYSIS

1.1.1 Objectives

The analyst needs certain requirements while designing and assembling the parts of the product. Those requirements are mentioned below.

To calculate,

i. Displacement at certain points;

ii. Stress distribution;

iii. Natural frequencies;

iv. Critical buckling loads;

v. Vibrations;

vi. Pressure, velocity and temperature distribution;

vii. Crack growth, residual strength and fatigue life.

1.1.2. Methods of Engineering Analysis

There are three different approaches to achieve the above mentioned objectives. they are

1. Experimental methods.
2. Analytical methods.
3. Numerical methods or approximate methods.

In this method, prototypes can be used. If we want to change the dimensions of the prototype, we have to disassemble the entire prototype and reassemble it and then testing should be carried out. It needs man power and materials. So, it is time consuming and costly process.

2. Analytical Methods or Theoretical Analysis

In these methods, problems are expressed by mathematical differential equations. It gives quick and closed form solutions. It is used only for simple geometries and idealized support and loading conditions.

3. Numerical Methods

Analytical solutions can be obtained only for certain simplified situations. For problems involving complex material properties and boundary conditions, the engineer prefers numerical methods that gives approximate but acceptable solutions. The following three methods are coming under numerical solutions.

(i) Functional Approximation.

(ii) Finite Difference Method (FDM).

(iii) Finite Element Method (FEM).

(i) Functional Approximation:

- The classical methods such as Rayleigh-Ritz methods (variational approach) and Galerkin methods (weighted residual methods) are based on functional approximation but vary in their procedure for evaluating the unknown parameters.
- Rayleigh-Ritz method is useful for solving complex structural problems, encountered in finite element analysis.
- Weighted residual method is useful for solving non-structural problems.

(ii) Finite Difference Method (FDM):

- Finite difference method is useful for solving heat transfer fluid mechanics and structural mechanics problems. It is a general method. It is applicable to any phenomenon for which differential equation along with the boundary conditions are available. It works well for two dimensional regions with boundaries parallel to the coordinate axes.

- The starting point in the finite difference method is that the differential equation must be known before solving. After that, the region is subdivided into a convenient number of divisions. The differential equation is applied successively at the various points of the subdivided region, a set of simultaneous equations are generated which upon solving lead to approximate solution to the problem. This is the essence of finite difference method.
- This method is difficult to use when regions have curved or irregular boundaries and it is difficult to write general computer programs.

(iii) Finite Element Method (FEM) or Finite Element Analysis (FEA):

- Finite element method is a numerical method for solving problems of Engineering and mathematical physics.
- In this method, a body or a structure in which the analysis to be carried out is subdivided into smaller elements of finite dimensions called finite elements. Then the body is considered as an assemblage of these elements connected at a finite number of joints called 'Nodes' or Nodal points. The properties of each type of finite element is obtained, assembled together and solved as whole to get solution.
- In other words, in the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and combine them to obtain the solution of the whole body.
- Finite element method is used to solve physical problems involving complicated geometrics, loading and material properties which cannot be solved by analytical methods. This method is extensively used in the field of structural mechanics, fluid mechanics, heat transfer, mass transfer, electric and magnetic fields problems.

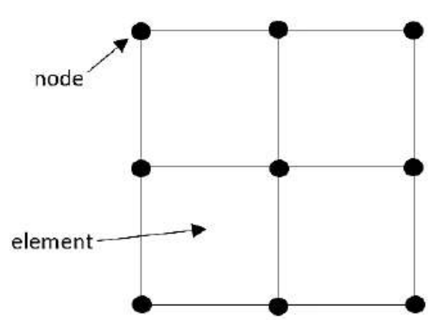


Figure 1.1 Shows the Finite Element Discretization

- Based on applications, the finite element problems are classified as follows:
  - (i) Structural problems
  - (ii) Non-structural problems.

(i) Structural problems:

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stress and strain in each element can be calculated.

(ii) Non-Structural problems

In non-structural problems, temperature or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow etc.., for each element can be calculated.

1.2 HISTORICAL BACKGROUND OF FEM

- Basis ideas of the finite element analysis were developed by aircraft engineers in the early 1940s. These were primarily the matrix methods of analysis.
- The modern development of the finite element method began in the year of 1945 in the field of structural engineering with the work by Hrennikoff.
- IN 1947 Levy introduced the flexibility of force method and in 1953 he suggested stiffness method which could be a promising alternative for use in analyzing statically redundant aircraft structures.
- By using energy principles, Argyris and Kelsey developed matrix structural analysis methods in 1954. This development illustrated the important role that energy principles would play in the finite element method.
- The term finite element was first introduced by Clough in 1960 in the plane stress analysis and he used both triangular and rectangular elements in that analysis.
- Most of the finite element work upto early 1960s dealt with small strains and small displacements, elastic material behavior and static loadings. In 1961, Turner considered large deflection and thermal analysis problems. In 1962, Gallagher introduced material no—linearities problems, whereas buckling problems were initially treated by Gallagher and padlog in 1963. In 1968, Zinkiewicz extended the method to visco elasticity problems.

- Weighted residual methods was first introduced by Szabo and Lee in 1969 for structural analysis and then by Zinkiewicz and parekh in 1970 for transient field problems.
- During the decades of the 1960s, and 1970s, the finite element method was extended to applications in shell bending, plate bending, heat transfer analysis, fluid flow analysis and general three dimensional problems in structural analysis.
- From the early 1950s to present, enormous advances have been made in the application of finite element method to solve complicated engineering problems. It is curious to mote that the present day finite element method does not have its root in one discipline. The mathematicians continue to put the finite element method on sound theoretical ground whereas the engineers continue to find interesting extensions in various branches of engineering. These concurrent developments have made the finite element method as one of the most powerful approximate methods.

1.3 GENERAL STEPS OF THE FINITE ELEMENT ANALYSIS

- This section presents the general procedure of finite element analysis. For simplicity's sake, we will consider only the structural problems.
- The following two general methods are associated with the finite element analysis.
  (i) Force method.
  (ii) Displacement or stiffness method.
- In force method, internal forces are considered as the unknowns of the problem. In displacement or stiffness method, displacement of the nodes are considered as the unknowns of the problem.
- Among these two approaches, displacement method is more desirable because its formulation is simpler for most structural analysis problems. So, a vast majority of general purpose finite element programs have used the displacement formulation for solving structural problems.
- We now present the steps along with explanations used in the finite element method formulation.

Step 1: Discretization of structure

The art of subdividing a structure into a convenient number of smaller elements is known as discretization.

Smaller elements are classified as follows:

(i) One dimensional elements.

(ii) Two dimensional elements

(iii) Three dimensional elements

(iv) Axisymmetric elements.

(i) One dimensional elements:

A bar and beam elements are considered as one dimensional elements. The simplest line element also known as linear element has two nodes, one at each end as shown in Figure 1.2.

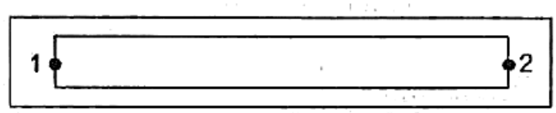


Figure 1.2 Bar Element.

(ii) Two dimensional elements:

Triangular and rectangular elements are considered as two dimensional elements. These elements are loaded by forces in their own plane. The simplest two dimensional elements have corner nodes as shown in Figure 1.3.

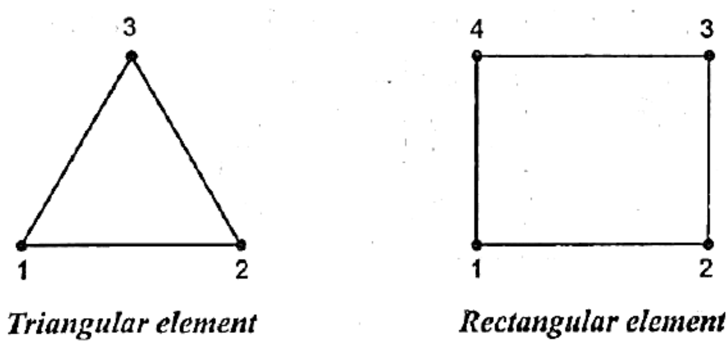


Figure 1.3 Simple Two Dimensional Elements

(iii) Three dimensional elements:

The most common three dimensional elements are tetrahedral and hexahedral (Brick) elements. These elements are used for three dimensional stress analysis problems. The simplest three dimensional elements have corner nodes as shown in Figure 1.4.

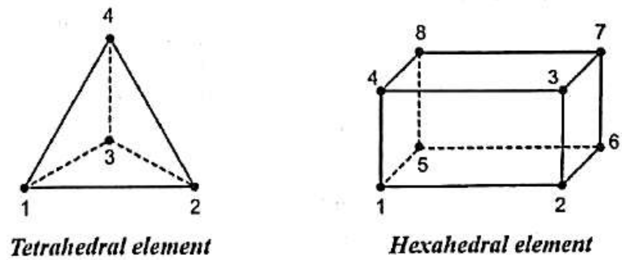


Figure 1.4 Simple Three Dimensional Elements

(iv) Axisymmetric elements:

The axisymmetric element is developed by rotating a triangle or quadrilateral about a fixed axis located in the plane of the element through 360^0. It is shown in Figure 1.5. when the geometry and loading of the problems are axisymmetric, these elements are used.

Step 2 : Numbering of Nodes and Elements

The nodes and elements should be numbered after discretization process. The numbering process is most important since it decide the size of the stiffness matrix and it leads the reduction of memory requirement. While numbering the nodes, the following condition should be satisfies.

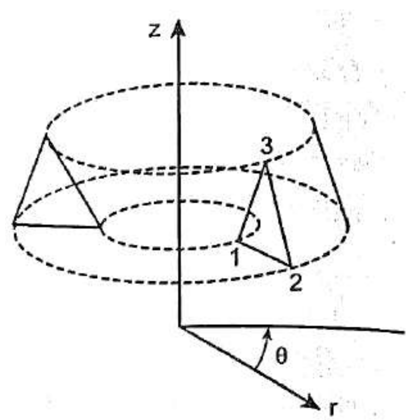


Figure 1.5

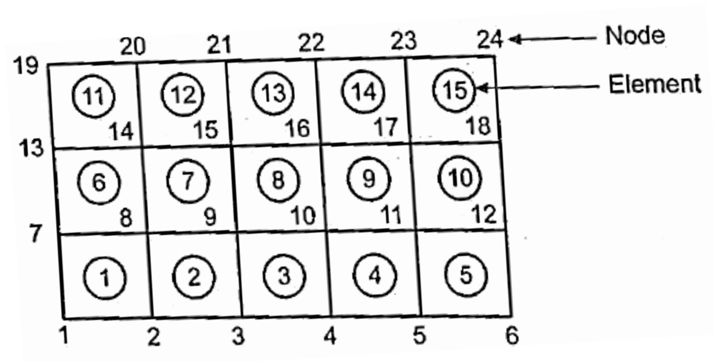
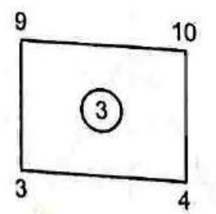


Figure 1.6 (a)

{Maximum node number} – {Minimum node number} = Minimum

It is explained in the Figure 1.6(a) and (b).

Longer Side Numbering Process:



[Note: Number with circle denotes element. Number without circle denotes node]

Considering element (3),

Maximum node number = 10

Minimum node number = 3

Difference = 7 …(1.1)

Shorter Side Numbering Process:

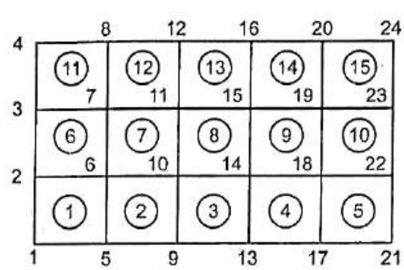
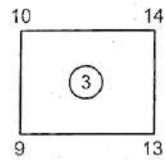


Figure 1.6 (b)

Considering the same element (3).



Maximum node number = 14

Minimum node number = 9

Difference = 5 …(1.2)

From equation (1.1) and (1.2), we came to know, shorter side numbering process is followed in the finite element analysis and it reduces the memory requirements.

Step 3: Selection of a Displacement Function or Interpolation Function

- It involves choosing a displacement function within each element. Polynomial of linear, quadratic and cubic form are frequently used as displacement functions because they are simple to work within finite element formulation.

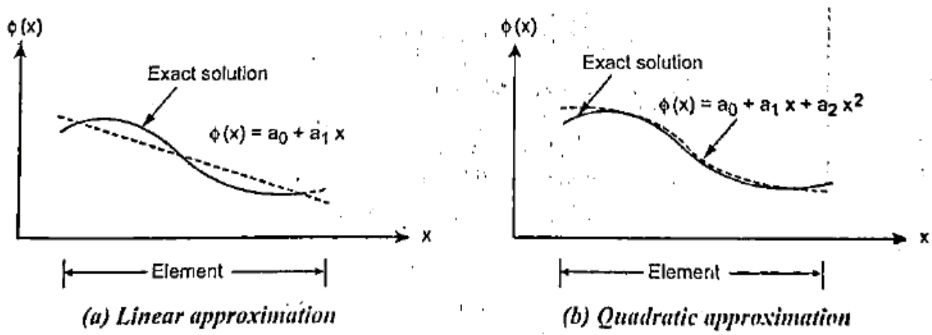


Figure 1.7 Polynomial Approximation in One Dimension

- The polynomial type of interpolation functions are mostly used due to the following reasons.
  1. It is easy to formulate and computerize the finite element equations.
  2. It is easy to perform differentiation or integration.
  3. The accuracy of the results can be improved by increasing the order of the polynomial.

  Figure 1.7 shows the polynomial approximation in one dimension.

Let us consider ϕ (x) is a fixed variable.

Case (i): Linear Polynomial:

One dimensional problem $\phi(x) = a_0 + a_1 x$

Two dimensional problem $\phi(x, y) = a_0 + a_1 x + a_2 y$

Three dimensional problem $\phi(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z$

Case (ii): Quadratic Polynomial:

One dimensional problem $\phi(x) = a_0 + a_1 x + a_2 x^2$

Two dimensional problem $\phi(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5\, xy$

Three dimensional problem $\phi(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 yz + a_9 xz$

Step 4: Define the Material Behavior by using Strain- Displacement and Stress – Strain Relationship

- Strain –Displacement and Stress-Strain relationships are necessary for deriving the equations for each finite element.
- In case of one dimensional deformation, the strain-displacement relationship is given by,

$$e = \frac{du}{dx} \qquad \text{... (1.3)}$$

Where, u → Displacement field variable along x direction.

e → Strain.

The stress-strain relationship is given by,

$$\sigma = E\,e \qquad \text{...(1.4)}$$

Where, σ → Stress in x direction.

E → Modulus of elasticity or Young's modulus.

Step 5 : Derivation of element stiffness matrix and equations:

The finite element equation is in matrix form as,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & \dots & \dots & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$$

In compact matrix from as,

$$\{ F^e\} = [k^e]\{ u^e\}$$

Where, e is a Element, { F } is the vector of element nodal forces, [k] is the element stiffness matrix and { u } is the element displacement vector.

This equation can be derived by any one of the following methods.

(i) Direct Equilibrium Method : This method is much easier to apply for line or one dimensional elements.

(ii) Variational Method : This method is most easily adaptable to the determination of element equations for complicated elements (i.e., element having large number of degrees of freedom) like axisymmetric stress element, plate bending element and two or three dimensional solid stress element.

(iii) Weighted Residual Method: This method is (Galerkin's method) useful for developing the element equations in thermal analysis problems. They are especially useful when a functional such as potential energy is not readily available.

Step 6: Assemble the element equations to obtain the global or total equations:

The individual element equations obtained in step 5 are added together by using a method of superposition i.e., direction stiffness method. The final assembled or global equation which is in the form of

$$\{ F\} = [K]\{u\} \qquad \dots (1.5)$$

Where, { F } → Global force vector.

[F] → Global stiffness matrix.

{ F } → Global displacement vector.

Step 7: Applying boundary conditions:

From equation (1.5), we know that, global stiffness matrix [K] is a singular matrix because its determination is equal to zero. In order to remove this singularity problem, certain boundary conditions are applied so that the structure remains in place instead of moving as a rigid body. The global equation (1.5) to be modified to account for the boundary conditions of the problem.

Step 8: Solution for the unknown displacements:

A set of simultaneous algebraic equations formed in step 6 can be written in expanded matrix form as follows:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \cdots & k_{4n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_n \end{Bmatrix}$$

These equation can be solved and unknown displacements { u } are calculated by using Gaussian elimination method or Gauss-Seidel method.

Step 9: Computation of the element strains and stresses from the nodal displacements, {u}:

In structural stress analysis problem, stress and strain are important factors. From the solution of displacement vector {u}, stress and strain value can be calculated.

In case of one dimensional deformation, the strain-displacement relationship is given by,

$$Strain, e = \frac{du}{dx} \qquad \text{[From equation (1.3)]}$$

$$= \frac{u_2 - u_1}{x_2 - x_1}$$

Where, u_1 and u_2 are displacement at node 1 and 2.

$x_2 - x_1$ = Actual length of the element.

From that, we can find the strain value.

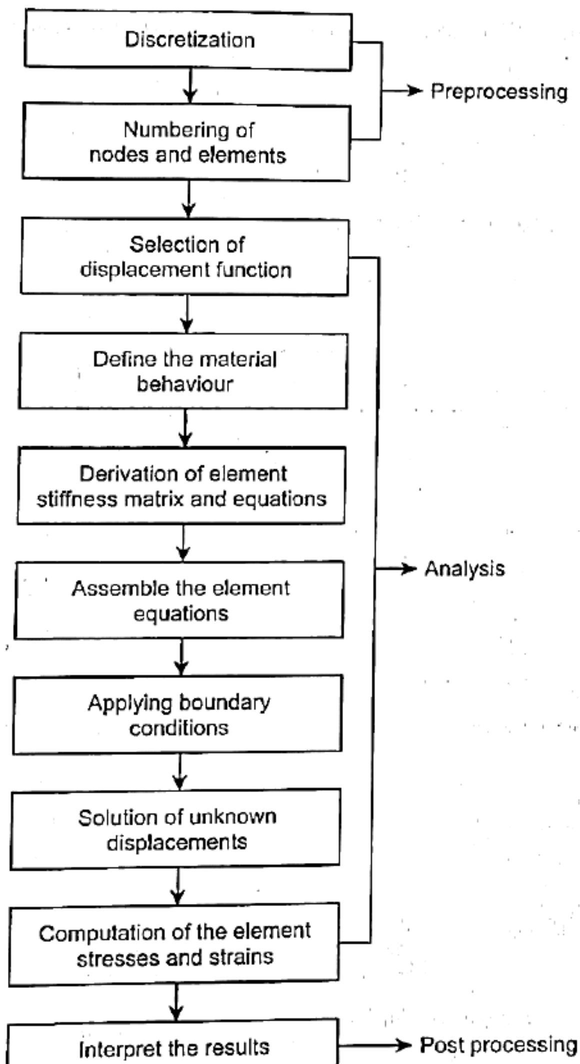
By knowing the strain, stress value can be calculated by using the relation, Stress, $\sigma = E\,e$ Where, $E \rightarrow$ Young's Modulus.

$e \rightarrow$ Strain.

Step 10: Interpret the results (Post Processing):

Analysis and evaluation of the solution results is referred to as post-processing. Post processor computer programs help the user to interpret the results by displaying them in graphical form.

Steps 1 to 10 are summarized as follows:



1.4 DISCRETIZATION

1.4.1 Introduction

In this chapter, we are going to learn about discretization, node, assembly, system etc. To make this easier to understand, let us compare these words with the parts over human body. Apart from flesh, our body consists of bones. They are hands, legs gingers, thigh bones, etc. these parts are connected together at different places, so that when movement takes place, we do not feel any pain. Nature has assembled in such a way that every human being is able to sustain certain amount of load without experiencing stain.

Similarly any structure like an automobile, ship, Aeroplane, etc., consists of several components assembled together.

Now let us study about 'Element'. The characteristics of an element are as follows:

(i) It is a small portion of a system

(ii) It has definite shape.

(iii) It should have minimum two nodes.

(iv) Nodes are placed where connection is made to another element.

(v) Loads act only at the nodes.

Examples:

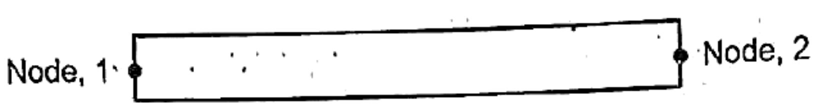
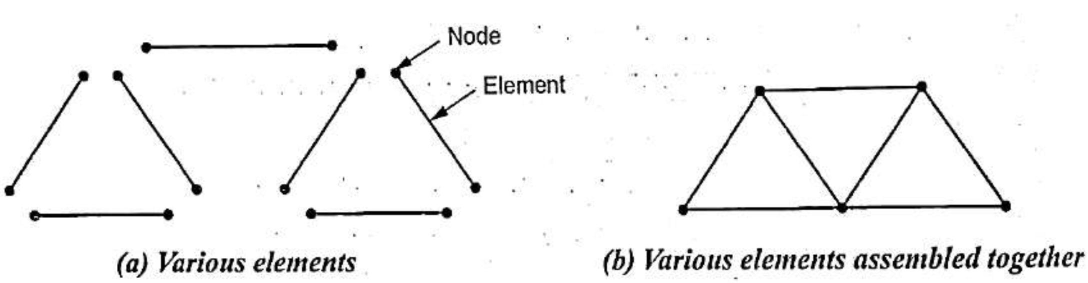


Figure 1.8 Truss element



(c) Imposition of boundary condition *(d) Structure carrying a load*

Figure 1.9

1.4.2 Discretization

The art of subdividing a structure into a convenient number of smaller components is known as Discretization. These smaller components are then put together. The process of uniting the various elements together is called Assemblage. The assemblage of such elements then represents the original body.

Discretization can be classified as follows:

(i) Natural.

(ii) Artificial (continuum).

1.4.3 Natural Discretization

In structural analysis, a truss is considered as a natural system. The various members of the truss constitute the elements. These elements are connected at various joints known as nodes.

Nodal Points: Each kind of finite element has a specific structural shape and is interconnected with the adjacent elements by nodal points or nodes.

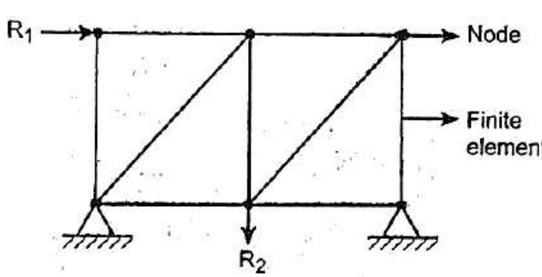


Figure 1.10 Natural discretization of truss

Nodal forces: The forces that act at each nodal point are called nodal forces.

Degrees of freedom: When the force or reaction act at nodal point, node is subjected to deformation. This deformation includes displacements, rotations, and/or strains. These are collectively known as degrees of freedom or simply we can say nodal displacement is called degrees of freedom.

In Fig.1.10, the truss consists of 9 elements and 6 nodes. There are four freely moving and two extreme constrained nodes. The truss is a natural system as there is no possibility either to increase or decrease the number of elements and the nodes.

1.4.4 Artificial Discretization (Continuum)

Continuum is generally considered to be a single mass of material as found in a forging, concrete dam, deep beam, plate and so on.

Unlike the truss element which is physically present in the truss, in a continuum, the following three elements exist only in our imagination.

1. Triangular element.
2. Rectangular element.
3. Quadrilateral element.

They are shown in Figure 1.11.

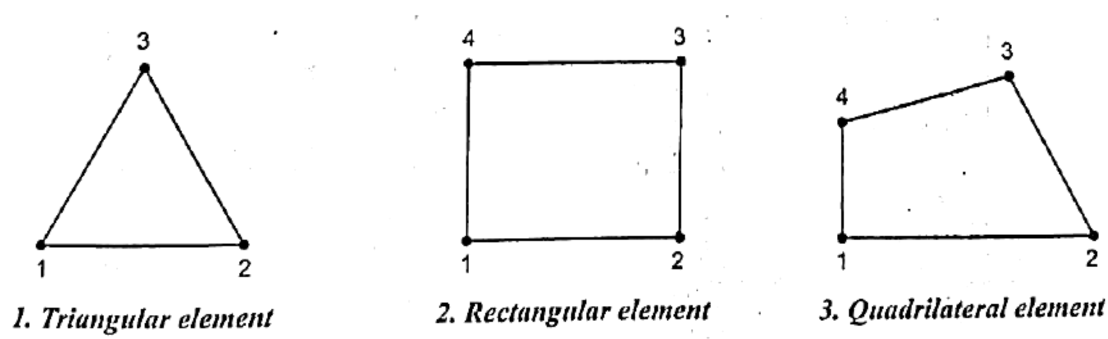


Figure 1.11.

Discretization using triangular element is shown in Figure 1.12. & Figure 1.13.

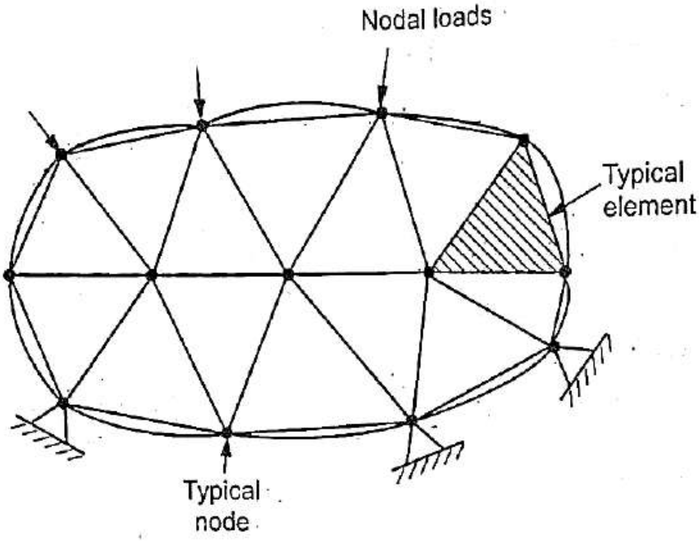


Figure 1.12. Discretization using triangular elements

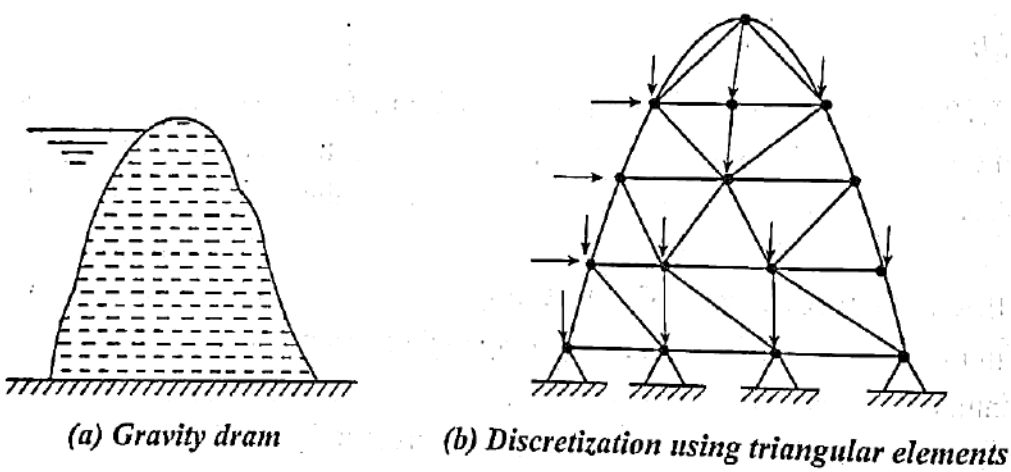


Figure 1.13.

Fig 1.14 shows a deep beam. In Fig.1.15, it is shown how it is discretized using simple rectangular elements.

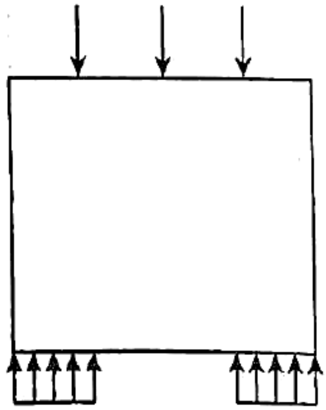


Figure 1.14. Deep beam

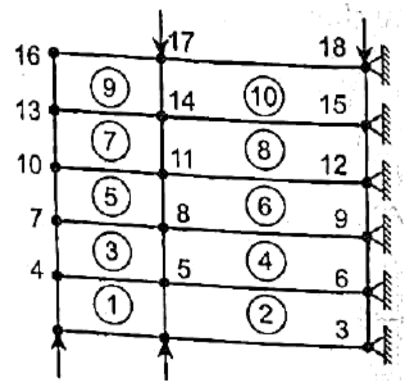
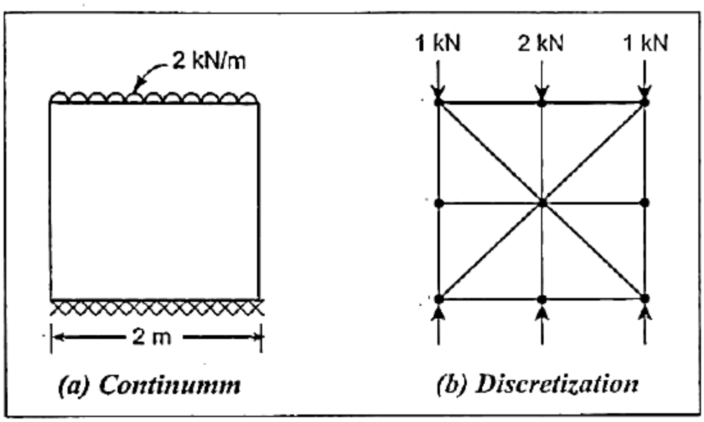


Figure 1.15. Deep beam discretization using Rectangular elements

Fig 1.16 (a) shows a planar continuum subjected to uniformly distributed load on the top.

Fig.1.16 (b), the continuum id discretized into eight triangular elements. The discretization shown is only one way. We can subdivide the continuum into triangular elements in a number of ways. Alternative way is shown in fig .1.17.



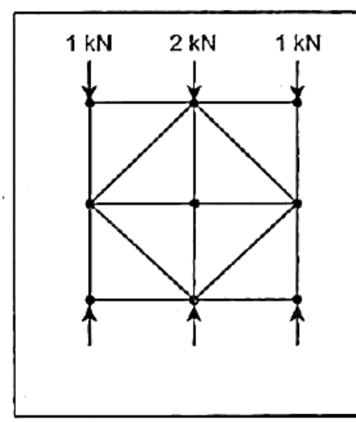


Figure 1.16. Figure 1.17. Alternative way of discretization

1.4.5 Discretization Process

The following points to be considered while analyzing the discretization process.

(i) Type of elements:

- The type of elements to be used will be evident from the physical problem.

- The structure, shown in Fig 1.18 is discretized by using line of bar elements.

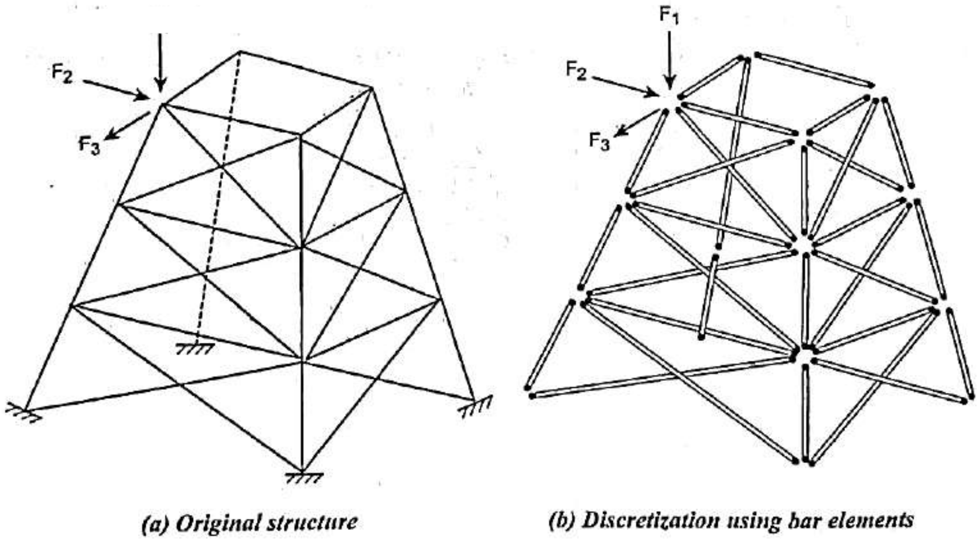


Figure 1.18.

- The finite element idealization can be done by using three dimensional rectangular element in stress analysis of short beam problem which is shown in Fig.1.19.

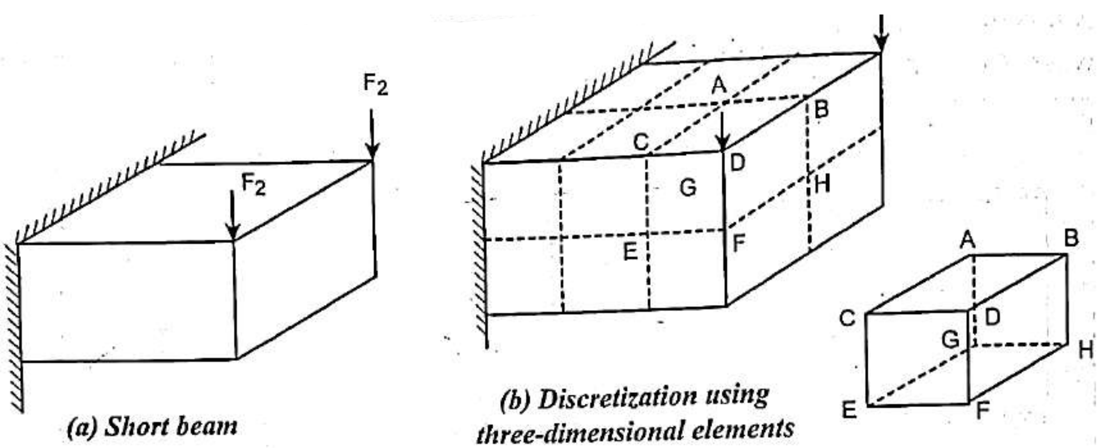


Figure 1.19. (a) Short beam (b) Discretization using three-dimensional elements

- A thin wall sheet shown in Fig.1.20 (a), which can be discretized by several types f elements as shown in Fig. 1.20(b).

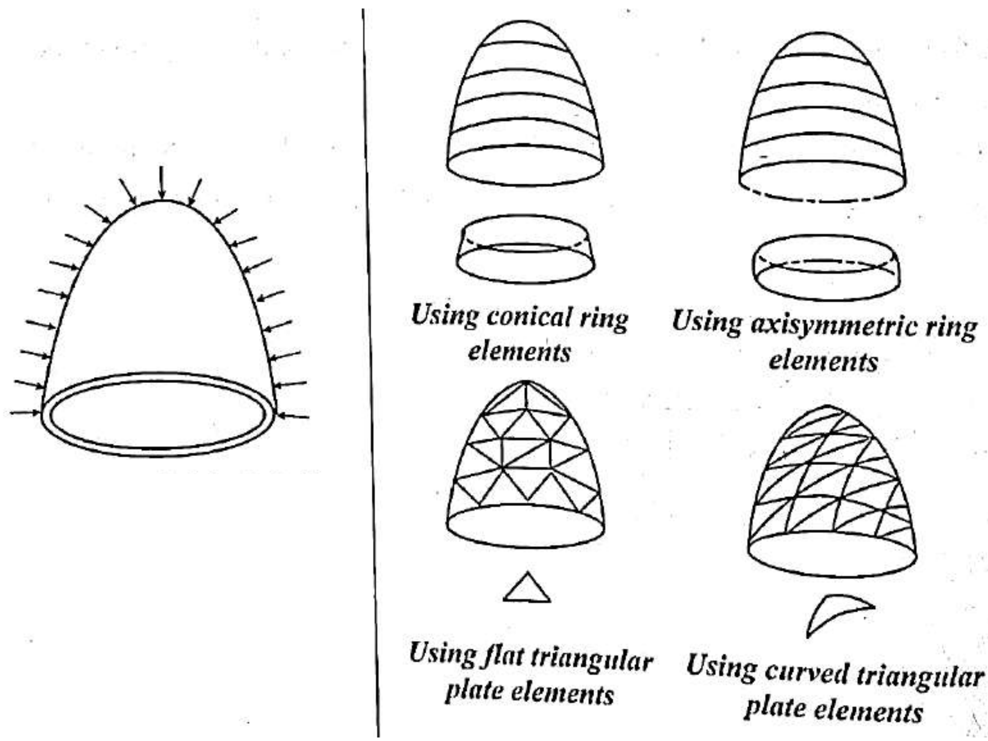


Figure 1.20. (a) Original shell (b) Discretization using different types of elements

- The choice of the element to be used for discretization depends upon the following factors.

  (i) Number of degrees of freedom needed.

  (ii) Expected accuracy.

  (iii) Necessary equations required.

- However, in certain problems, the given structure cannot be discretized by using only one type of elements. In such cases, we can use two or more types of elements for discretization.

  Example: Air craft wing.

(ii) Size of elements:

- The size of elements influences the convergence of the solution of the problem directly. So, it should be chosen with more care.

- If the size of the element is small, the final solution is more accurate. But the computational time for the smaller size element is more when compared to larger size element.
- Another characteristic related to the size of elements that affects the finite element problem solution is the " Aspect ratio" of the elements.
- Aspect ratio is defined as the ratio of the largest dimension of the element to the smallest dimension. The conclusion of many researchers is that the aspect ratio should be close to unity as possible. For a two dimensional rectangular element, the aspect ratio is conveniently defined as length to breadth ratio. Aspect ratio closer to unity yields better results.

(iii) Location of nodes:

- If the structure has no abrupt changes in geometric, load, boundary conditions and material properties, the structure can be divided into equal subdivisions. So, the spacing of the nodes are uniform.
- If there are any discontinuities in geometric, load, boundary conditions and material properties of the structure, nodes should be introduced at these discontinuities as shown in the following figures.

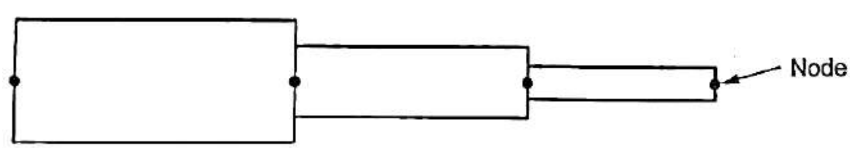


Figure 1.21. Geometric discontinuities

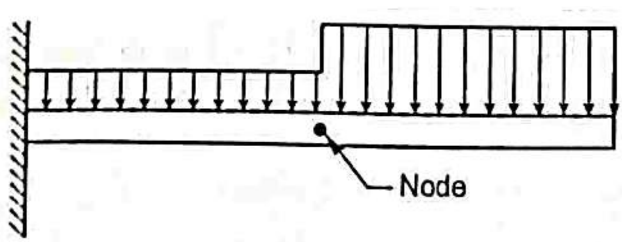


Figure 1.22. Discontinuity in loading

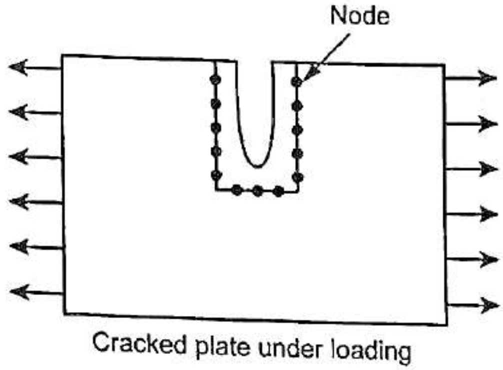


Figure 1.23. Discontinuity of boundary conditions

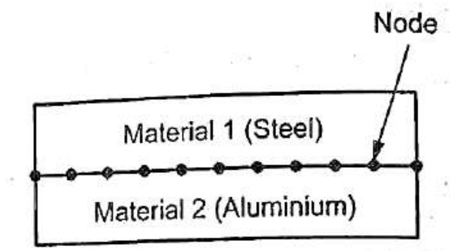


Figure 1.24. Material discontinuity

(iv) Number of elements:

The number of elements to be selected for discretization depends upon the following factors:

1. Accuracy desired.
2. Size of the elements.
3. Number of degrees of freedom involved.

- If the number of elements in the structure is increased, the final solution of the problem is expected to be more accurate. But the use of large number of elements involves a large number of degrees of freedom, it leads the storage problem in the available computer memory.

1.5 INITIAL VALUE AND BOUNDARY VALUE PROBLEMS

A differential equation along with subsidiary conditions on the unknown function and its c=derivatives, all given at the same value of the independent variable, constitutes an initial value problem. The subsidiary conditions are initial conditions. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a boundary value problem and the conditions are boundary conditions.

For Example:

- The problem $y'' + 2\,y' = e^x; y(\pi) = 1, y'(\pi) = 2$ is an initial-value problem, because both the subsidiary conditions are given at $x = \pi$.
- The problem $y'' + 2\,y' = e^x; y(0) = 1, y(1) = 1$ is an boundary-value problem, because the two subsidiary conditions are given at the different values $x = 0$ and $x = 1$.

1.6 PROBLEM BASED ON INITIAL VALUE PROBLEM

Example 1.1 Find the solution of the initial value problem.

$y' + y = 0; y(3) = 2$

Given : Differential equation, $y' + y = 0$

Boundary condition at $y(3) = 2$

Solution: Differential equation, $y' + y = 0$ ...(1)

$$\left.\begin{aligned}\text{Boundary condition at } y(3) &= 2\\ \Rightarrow x = 3, y &= 2\end{aligned}\right\} \quad \text{...(2)}$$

Using Auxiliary equation,

$$\lambda + 1 = 0 \qquad \left[\because y' = \frac{d}{dx} = \lambda\right]$$

$$\lambda = -1 \qquad \text{...(3)}$$

We know that, complementary function or characteristic function,

$$y(x) = c_1 e^{-x} \qquad \text{... (4)}$$

Applying the boundary condition (2) in equation (4),

$$y(3) = c_1 e^{-3}$$

$$2 = c_1 e^{-3} \qquad [\because y(3) = 2]$$

$$c_1 = 2\,e^3$$

By substituting equation (5) in equation (4)

$$y(x) = 2e^3 e^{-x}$$

As the solution of the initial-value problem.

Result: $y(x) = 2e^3\, e^{-x}$, as the solution of the initial-value problem.

Example 1.2

Find a solution of the initial-value problem $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2\,y = 0$, Boundary conditions y(0) =2, y'(0) =5.

Given:

Differential equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

Boundary conditions are $y(0) = 2\,, y'(0) = 5$

Solution: differential equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \qquad \ldots(1)$$

Boundary conditions are $y(0) = 2\,, y'(0) = 5$ ...(2)

Using auxiliary equation, $\lambda^2 + \lambda - 2 = 0$

$$(\lambda - 1)(\lambda + 2) = 0$$

$$\Rightarrow \qquad \lambda_1 = 1,\ \lambda_2 = -2 \qquad \ldots(3)$$

We know that, complementary function, $y(x) = A\, e^{\lambda}1x + Be^{\lambda\, 2x}$

Put $\lambda_1 = 1, \lambda_2 = -2$ in the above equation,

$$y(x) = A\, e^{x} + Be^{-2x} \qquad \ldots(4)$$

$$y'^{(x)} = A\, e^{x} - 2Be^{-2x} \qquad \ldots(5)$$

Applying the boundary conditions (2) in equation (4) and (5),we get

$$y(0) = 2 \Rightarrow x = 0,\ y = 2$$

$$y(0) = Ae^{0} + Be^{0}$$

$$2 = A + B$$

$$A + B = 2 \qquad \ldots(6)$$

Similarly, y'(0) = x = 0, y = 5

$$y'^{(0)} = Ae^0 - 2\,Be^0$$

5 = A - 2 B

A – 2 B = 5 …(7)

Solving the equation (6) and (7),

A + B = 2

A -2 B = 5

(-) (+) (-)

3 B = -3

B = -1

Substitute the B = -1 value in equation (6),

A -1 = 2

A = 2 + 1

A = 3

By substituting A and B values in equation (4),

$$y(x) = 3\,e^x - e^{-2x}$$

Result: General solution $y(x) = 3\,e^x - e^{-2x}$

1.7 PROBLMES SOLVED ON BOUNDARY-VALUE PROBLEM

EXAMPLE 1.3

Find a solution of a boundary-value problem $y'' + y = 0\, with\, y\,(0) = 0\, and\, y(\pi/6) = 4.$

Given: Differential equation, y" + y = 0

Boundary conditions are y(0) = 0, y(π/6) = 4

Solution: Differential equation, y" + y = 0 …(1)

Boundary conditions are y(0) = 0, y (π/6) = 4

Using auxiliary equation, $\lambda^2 + 1 = 0$

$$\left[\because y' = \frac{d}{dx} = \lambda;\ y" = \frac{d^2}{dx^2} = \lambda^2\right]$$

$$\lambda^2 = -1$$

$$\lambda = \sqrt{-1}$$

$$\lambda = \pm i$$

$$\lambda = \propto \pm i\beta \qquad [Here, \propto = 0, \beta = 1]$$

We know that, complementary functions are,

$$y(x) = e^{\propto x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

$$y(x) = e^0[c_1 \cos x + c_2 \sin x]$$

$$y(x) = c_1 \cos x + c_2 \sin x \qquad \text{... (2)}$$

Applying boundary conditions in equation (2),

$$y(0) = 0 \Rightarrow y(0) = c_1 \cos(0) + c_2 \sin(0)$$

$$0 = c_1 + 0 \qquad [\because \cos 0 = 1, \sin 0 = 0]$$

$$c_1 = 0$$

similarly,

$$y(\pi/6) = 4 \Rightarrow y\ (\pi/6) = c_1 \cos(\pi/6) + c_2 \sin(\pi/6)$$

$$4 = c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right)$$

$$put\ c_1 = 0, c_2\left(\frac{1}{2}\right) = 4$$

$$c_2 = 8$$

Substitute the $c_1 and\ c_2$ values in equation (2)

$$y(x) = 0 + 8 \sin x$$

$$y(x) = 8 \sin x$$

Result : General equation, $y(x) = 8 \sin x$

Example 1.4

Find a solution of a boundary-value problem $y'' + 4y = 0 \; with \; y(\pi/8) = 0, y(\pi/6) = 1$.

Given : Differential equation, y” + 4y = 0 …(1)

Boundary conditions are y(π/8) = 0 and y(π/6) = 1 …(2)

Solution: Differential equation, y” + 4y = 0 …(1)

Boundary conditions are y(π/8) = 0 and y(π/6) = 1

Using auxiliary equation, $\lambda^2 + 4 = 0$

$$\lambda^2 = -4$$

$$\lambda = \sqrt{-4}$$

$$\lambda = \pm 2i$$

$$\lambda = \propto \pm i\beta \qquad [\because Here, \propto = 0, \beta = 2]$$

We know that, complementary functions,

$$y(x) = e^{\propto x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

$$y(x) = e^{0}[c_1 \cos 2x + c_2 \sin 2x]$$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x \qquad \ldots(3)$$

Applying first boundary conditions in equation (3),

y(π/8) = 0

$$\Rightarrow \qquad y(\pi/8) = c_1 \cos 2(\pi/8) + c_2 \cos 2(\pi/8)$$

$$y(\pi/8) = c_1 \cos (\pi/4) + c_2 sin(\pi/4)$$

$$0 = c_1\left(\frac{1}{2} \times \sqrt{2}\right) + c_2\left(\frac{1}{2} \times \sqrt{2}\right)$$

$$c_1\left(\frac{\sqrt{2}}{2}\right) + c_2\left(\frac{\sqrt{2}}{2}\right) = 0$$

$$(or)\; c_1\left(\frac{1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right) = 0$$

Applying second boundary conditions in equation (3),

$$y(\pi/6) = 1$$

$$\Rightarrow \quad y(\pi/6) = c_1 \cos\ 2(\pi/6) + c_2 \sin 2(\pi/6)$$

$$1 = c_1 \cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3}$$

$$c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = 1$$

Solving the equation (4) and (5),

Equation (4) $\Rightarrow$

$$c_1\left(\frac{1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right) = 0$$

Equation (5) $\Rightarrow$

$$c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = 1$$

Get, $$c_1 = 2.732$$

Substitute the c_1 value in equation (5)

$$2.732\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = 1$$

$$c_2 = -2.732$$

From these c_1 and c_2 value, substitute the equation (3)

$$y(x) = 2.732 \cos 2\,x + (-2.732) \sin 2\,x$$

$$y(x) = 2.732\,[\cos 2\,x - \sin 2\,x]$$

Result: General equation, $y(x) = 2.732\,[\cos 2\,x - \sin 2\,x]$

1.8 EIGEN VALUE PROBLEM [BOUNDARY VALUE PROBLEM]

When applied to the boundary- value problem, has the form

$$y'' + P(x+\lambda)y' + Q(x+\lambda)y = 0$$

Non-trivial solutions may exit for certain values of λ. Those values of λ for which no-trivial solutions do exists are called eigen values, the corresponding non-trivial solutions are called eigen functions.

For example: for the axial vibration of bar, to find u (x) and λ that satisfies the partial differentiation equation and boundary conditions are,

$$A\,y'' + \lambda\,y' = 0, for\ 0 < x < L$$

$$y(0) = 0, y'(L) = 0$$

1.9 PROBLMES SOLVED ON EIGEN VALUE PROBLEM [BOUNDARY VALUE PROBLEM]

Example 1.5

Find a eigen values and eigen function $y'' + 4\lambda y' + \lambda^2 y = 0$; with boundary conditions are $y(0) = 0, y(1) + y'(1) = 0$

Given: Differential equation,

$$y'' - 4\lambda\,y' + 4\lambda^2 y = 0$$

Boundary conditions are, y(0) = 0

y(1) + y'(1) = 0

Solution: Given differential equation,

$$y'' - 4\lambda\,y' + 4\lambda^2 y = 0 \qquad ...(1)$$

Boundary conditions are, y(0) = 0

y(1) + y'(1) = 0 ...(2)

The auxiliary equation is,

$$m^2 - 4\,\lambda\,m + 4\,\lambda^2 = 0 \qquad ...(3)$$

$$\Rightarrow \qquad = (m - 2\,\lambda)(m - 2\,\lambda) = 0$$

$$\Rightarrow \qquad = (m - 2\,\lambda) = 0\ (m - 2\,\lambda) = 0$$

$$\Rightarrow \qquad = m_1 = 2\,\lambda, m_2 = 2\,\lambda$$

We know that, complementary function is,

$$y(x) = c_1 e^{m1^x} + c_2 x\, e^{m\,2^x} \quad \ldots(4)$$

$$y(x) = c_1 e^{2\lambda^x} + c_2 x\, e^{2\,\lambda^x}$$

Differentiate with respect to "x" in equation (4),

$$y'(x) = 2\,\lambda\, c_1\, e^{2\,\lambda\, x} + c_2[\,x\, 2\,\lambda\, e^{2\lambda x} + e^{2\,\lambda\, x}]$$

$$y'(x) = 2\,\lambda\, c_1\, e^{2\,\lambda\, x} + c_2\left[\,x\, 2\,\lambda\, e^{2\lambda x} + e^{2\,\lambda\, x}\right] \quad \ldots(5)$$

Applying first boundary condition in equation (5),

y(0) = 0, x = 0, y = 0

$$\Rightarrow \quad y(0) = c_1 e^0 + c_2(0)e^0$$

$$c_1 = 0$$

Applying second boundary conditions in equation (5),

$$y(1) + y'(1) = 0$$

We get, $$y(1) = c_1 e^{2\,\lambda} + c_2 e^{2\,\lambda}$$

$$y'^{(1)} = 2\,\lambda\, c_1 e^{2\,\lambda} + c_2(2\,\lambda\, e^{2\,\lambda} + \ e^{2\,\lambda})$$

We know that,

$$\Rightarrow \quad c_1 e^{2\,\lambda} + c_2 e^{2\,\lambda} + 2\,\lambda\, c_1 e^{2\,\lambda} + c_2(2\,\lambda\, e^{2\,\lambda} + \ e^{2\,\lambda}) = 0$$

$$c_1(1 + 2\,\lambda) + c_2(2 + 2\,\lambda) = 0$$

If now follows that $C_1 = 0$ and either $C_2 = 0$ (or) $\lambda = -1$

The choice of $C_2 = 0$.

The result in the trivial solution y = 0.

The choice of $\lambda = -1$.

The result in the non-trivial solution, i.e., $y = c_2 e^{-2\,x}$, where $c_2 \Rightarrow$ arbitrary

Thus the boundary value problem has eigen value $\lambda = -1$ and the eigen function

$$y = c_2 e^{-2\,x}$$

Result: Eigen value and eigen functions, $y = c_2 e^{-2\,x}$

Example 1.6

Find a eigen values and eigen function $y'' + 4\lambda y' + 4\lambda^2 y = 0$; with boundary conditions are $y'^{(1)} = 0, y(2) + 2y'(2) = 0.$

Given : Differential equation,

$$y'' - 4\lambda y' + 4\lambda^2 y = 0$$

Boundary conditions are, $y'(1) = 0, y(2) + 2y'(2) = 0.$

Solution: Differential equation,

$$y'' + 4\lambda y' + 4\lambda^2 y = 0 \qquad \text{... (1)}$$

Boundary conditions are, $y'(1) = 0$

$$y(2) + 2y'(2) = 0 \qquad \text{...(2)}$$

The auxiliary equation is, $m^2 - 4\lambda m + 4\lambda^2 = 0$ ... (3)

$$(m - 2\lambda)(m - 2\lambda) = 0$$

$$m_1 = 2\lambda, m_2 = 2\lambda$$

We know that, complementary functions are,

$$y(x) = c_1 e^{2\lambda x} + c_2 x\, e^{2\lambda x} \qquad \text{... (4)}$$

Differentiate with respect to "x" in equation (4),

$$y'(x) = 2\lambda c_1 e^{2\lambda x} + c_2\left[x\, 2\lambda e^{2\lambda x} + e^{2\lambda x}\right] \qquad \text{... (5)}$$

Applying first boundary condition in equation (5), we get,

$$y'(1) = 0$$

$$\Rightarrow \quad y'(1) = 2\lambda c_1 e^{2\lambda} + c_2\left(2\lambda e^{2\lambda} + e^{2\lambda}\right) = 0$$

$$\Rightarrow \quad 2\lambda c_1 e^{2\lambda} + c_2[2\lambda e^{2\lambda} + e^{2\lambda}] = 0$$

$$\Rightarrow \quad 2\lambda c_1 + c_2\,[2\lambda + 1] = 0 \qquad \text{... (6)}$$

Applying second boundary conditions in equation (5) and (4),

$$y(2) + 2y'(2) = 0$$

$$\Rightarrow \quad y(2) = c_1 e^{4\lambda} + 2c_2 e^{4\lambda}$$

$\Rightarrow \qquad y'(2) = 2\,\lambda\, c_1 e^{4\,\lambda} + c_2(4\,\lambda\, e^{4\,\lambda} + e^{4\,\lambda})$

Adding both equations,

$$\Rightarrow \qquad c_1 e^{4\lambda} + 2c_2\, e^{4\lambda} + 2\left[\lambda\, c_1 e^{4\,\lambda} + c_2\left(4\,\lambda\, e^{4\,\lambda} + e^{4\lambda}\right)\right] = 0$$

$$\Rightarrow \qquad c_1 e^{4\lambda} + 2c_2\, e^{4\lambda} + 4\lambda\, c_1 e^{4\,\lambda} + 2\, c_2\left(4\,\lambda\, e^{4\,\lambda} + e^{4\lambda}\right) = 0$$

$$\Rightarrow \qquad c_1 e^{4\lambda} + 2c_2\, e^{4\lambda} + 4\,\lambda\, c_1 e^{4\,\lambda} + 8\,\lambda\, c_2\, e^{4\,\lambda} + 2\, c_2\, e^{4\lambda} = 0$$

$$\Rightarrow \qquad c_1 + 2c_2 + 4\,\lambda\, c_1 + 8\,\lambda\, c_2 + 2\, c_2 = 0$$

$$\Rightarrow \qquad (1 + 4\,\lambda)c_1 + (8\,\lambda + 4)c_2 = 0 \qquad \ldots(7)$$

Solving equations (6) and (7),

$$\Rightarrow \qquad 2\,\lambda\, c_1 + c_2(\,2\,\lambda + 1\,) = 0$$

$$\Rightarrow \qquad (1 + 4\,\lambda)\, c_1 + (8\,\lambda + 4)\, c_2 = 0$$

Set determination is equal to zero,

$$\begin{vmatrix} 2\,\lambda & 1 + 2\,\lambda \\ 1 + 4\,\lambda & 4 + 8\,\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad 2\,\lambda\,(4 + 8\,\lambda) - (1 + 2\,\lambda)(1 + 4\,\lambda) = 0$$

$$\Rightarrow \qquad 8\,\lambda\,(1 + 2\,\lambda) - (1 + 2\,\lambda)(1 + 4\,\lambda) = 0$$

$$\Rightarrow \qquad (1 + 2\,\lambda) - [8\,\lambda - (1 + 4\,\lambda)] = 0$$

$$\Rightarrow \qquad (1 + 2\,\lambda) - (1 + 4\,\lambda) = 0$$

$$\Rightarrow \qquad (1 + 2\,\lambda) = 0 \; and \; (4\,\lambda - 1) = 0$$

$$\lambda_1 = -\frac{1}{2} and\; \lambda_2 = \frac{1}{4}$$

When, $\lambda_1 = -\frac{1}{2} and\; \lambda_2 = \frac{1}{4}$ the result has non-trivial solution.

It follows that eigen values are $\lambda_1 = -\frac{1}{2} and\; \lambda_2 = \frac{1}{4}$ and the corresponding eigen functions are,

$$y_1 = c_2 x\, e^{-x} \text{and}$$

$$y_2 = c_2(-3 + x)\, e^{x/2}$$

Result: Eigen value, $\lambda_1 = -\frac{1}{2} and\ \lambda_2 = \frac{1}{4}$

Eigen functions, $y_1 = c_2 x\, e^{-x} and$

$y_2 = c_2(-3 + x)\, e^{x/2}$

1.10 SOLUTION OF EIGEN VALUE PROBLEMS [MATRIX APPROACH]

There are three methods to solve eigen value problems, They are,

1. Determinant methods
2. Transformation methods
3. Vector iteration methods

1.10.1 Determinant Methods

These methods are primarily based on the equations,

$$\{[K] - \lambda\, [m]\}\, \{u\} = 0$$

If the eigen vector is non-trivial, the required condition is,

$$\det|[K] - \lambda\, [m]| = 0$$

$$\Rightarrow \quad |[K] - \lambda\, [m]| = 0 \qquad \dots (1.6)$$

Trial value of λ is taken and the determinant | [K] - λ [m] | = 0 is computer. The curve is drawn by taking several trial values.

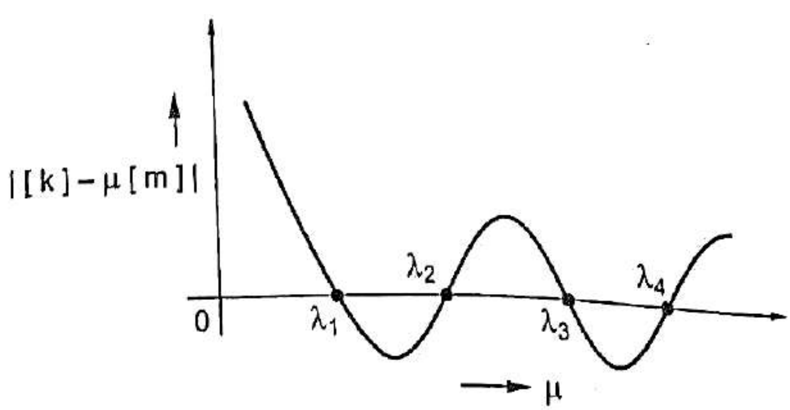


Figure 1.25. Determinant-based Method

Due to heavy computational cost and several iterations are required to determine all the eigen values, the determinant based methods are not implemented in practice.

1.10.2 Transformation Methods

This method is used to transform the eigen value problems,

Let, $$[K]\{u\} = \lambda\{u\} \qquad \text{... (1.7)}$$

Transform [K] into a diagonal matrix by using a series of matrix transformations,

$$[K] = [T]^T[K][T] \qquad \text{... (1.8)}$$

Where, [T] is the transformation matrix, which is usually an orthogonal matrix.

$i.e.,$ $$[T]^T = [T]^{-1}$$

When we transform [K] completely into diagonal matrix, then the elements on the diagonals are considered as eigen values,

$$[T]^T[K][T] = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} \qquad \text{... (1.9)}$$

Where, λ_1, λ_2 and λ_3 are eigen values.

1.10.3 Vector iteration Methods

- Vector iteration methods are normally available in many commercial finite element software packages.
- In this method, trial eigen vector is assumed and repeated matrix manipulation is performed to compute the desired eigen vector.

1.11 SOLVED PROBLEMS ON EIGEN VALUES [MATRIX APPROACH]

Example 1.7 Find the eigen values of $\begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$

Solution:

Step 1: To find characteristic equation, Let the given matrix be $A = \begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \dots (1)$$

Where, a_1= Sum of leading diagonal elements

= 4 + 10 = 13

$$a_1 = 1 \qquad \dots (2)$$

a_2 = Sum of minors of the leading diagonal elements

$$= \begin{vmatrix} 10 & 4 \\ -30 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -10 \\ 6 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -20 \\ -2 & 10 \end{vmatrix}$$

= -130 + 120 – 52 + 60 + 40 – 40

$$a_2 = -2 \qquad \dots (3)$$

$$a_3 = |A| = \begin{vmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{vmatrix}$$

= 4[-130+ 120] +20 [26-24] – 10[60 -60]

= -40 + 40 + 0

$$a_3 = 0 \qquad \dots (4)$$

Substitute the a_1, a_2 *and* a_3 values in equation (1),

$$\lambda^3 - \lambda^2 - 2\,\lambda = 0$$

Step 2: To find eigen values:

$$\lambda^3 - \lambda^2 - 2\,\lambda = 0$$

$$\lambda(\lambda^2 - \lambda - 2) = 0$$

$When\ \lambda = 0,$ $\qquad \lambda^2 - \lambda - 2 = 0$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

= 2 or -1

∴ Eigen values are $\lambda = 0, -1, 2$

Result: Eigen values, $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 2$.

1.12 WEIGHTED RESUDUAL METHODS

1.12.1 Introduction

The method of weighted residuals is a powerful approximate procedure applicable to several problems. For structural problems, potential energy functional can be easily formed, so, Rayleigh-Ritz method is used. On the other hand, for non-structural problems, the differential equation of the phenomenon can be easily formulated. For such type of problems, the method of weighted residuals becomes very useful. There are many types of weighted residuals, of them four are very popular. They are.

(i) Point collocation method.

(ii) Subdomain collocation method.

(iii) Least squares method.

(iv) Galerkin's method.

Among these four methods, the Galerkin approach has the widest choice and is used in finite element analysis.

1.12.2. General Procedure

Our interest is to find *y*, which is the solution for the differential equation. If it is not possible to find a solution, we assume an approximate function for y. When we substitute the approximate solution in the differential equation, we can get residual and that residual can be expresses as,

$$R(x_i; a_1, a_2, a_3) = 0$$

Where a_1, a_2 are unknown parameters present in assumed trial function.

The assumed trial function can be expressed as follows:

$$y = f(x; a_1, a_2, a_3, \dots \dots, a_n)$$

Trial function y must exactly satisfy the boundary conditions.

The method of weighted residuals needs the parameters $a_1, a_2, a_3, \dots \dots, a_n$ to be determined by satisfying the following equation.

$$\int_D w_i R(x; a_1, a_2, a_3, \ldots, a_n) dx = 0 \qquad \ldots (1.10)$$

Where, w_i is a function of x and known as weighting function.

Dis a domain, R is a residual.

1.12.3. Point Collocation Method

In the collocation method, also called point collocation, residuals are set to zero at n different locations X_i, and the weighting function w_i is denoted as $\delta(x- x_i)$

$$\Rightarrow \qquad w_i = \delta(x - x_i)$$

Substituting $w_i$value in equation (1.10),

$$\int_D \delta(x - x_i)\, R(x; a_1, a_2, a_3, \ldots, a_n) dx = 0 \qquad \ldots (1.11)$$

The $x_i's$ are referred to as collocation points and are selected by the discretion of the analyst.

In equation (1.11), term $\int_D \delta(x - x_i) = 1$

$$So, \qquad R(x; a_1, a_2, a_3, \ldots, a_n) = 0$$

1.12.4 Subdomain Collocation Method

In this method, the weighting functions (w_i) are chosen to be unity over a portion of the domain and zero elsewhere. It is given as follows:

$$w_1 = \begin{cases} 1 \text{ for } x \text{ in } D_1 \\ 0 \text{ for } x \text{ not in } D_1 \end{cases}$$

$$w_2 = \begin{cases} 1 \text{ for } x \text{ in } D_2 \\ 0 \text{ for } x \text{ not in } D_2 \end{cases}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$w_n = \begin{cases} 1 \text{ for } x \text{ in } D_n \\ 0 \text{ for } x \text{ not in } D_n \end{cases}$$

Where D is a domain.

1.12.5 Least Squares Method

In this method, the integral of the weighted square of the residual over the domain is required to be minimum.

$$i.e., \qquad I = \int_D [R(x; a_1, a_2, a_3, \dots \dots . a_n)]^2 dx = \text{minimum}$$

$$where, \qquad I = f(a_1, a_2, \dots \dots a_n)$$

The requirement is

$$\frac{\partial I}{\partial a_i} = 0, i = 1, 2, 3, \dots . . n$$

1.12.6 Galerkin's Method

In this method, the trial function, $N_i(x)$, itself is considered as the weighting functions; that is,

$$w_i = N_i(x)$$

Substitute w_i value in equation (1.10),

$$\Rightarrow \qquad \int_D N_i(x) R(x; a_1, a_2, \dots . . a_n) dx = 0 \qquad \dots (1.12)$$

i = 1, 2, 3, ….. n

1.12.7 Solved Problems – Weighted Residual Method

Example 1.8

The following differential equation is available for a physical phenomenon $\mathrm{A\,E}\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + \mathrm{q_0} = 0$ with the boundary conditions

$$y(0) = 0$$

$$\left.\frac{dy}{dx}\right|_{X=L} = 0$$

Find the value of f(x) using the weighted residual method.

Given: Differential equation, $A\,E\frac{d^2y}{dx^2} + q_0 = 0$

Boundary conditions are y(0) = 0

$$\left.\frac{dy}{dx}\right|_{X=L} = 0$$

To find: *f(x)*

Solution: Assume a trial solution.

$$Let \qquad y(x) = a_0 + a_1x + a_2x^2 \qquad \ldots.(1)$$

Apply first boundary condition, i.e., substitute x =0 and y =0 in equation (1).

$$(1) \Rightarrow \; 0 = a_0 + 0 + 0$$

$$\Rightarrow a_0 = 0$$

Apply second boundary condition,

$$y(x) = a_0 + a_1x + a_2x^2$$

$$\frac{dy}{dx} = a_1 + 2\,a_2x$$

$$At\ x = L, \qquad \frac{dy}{dx} = 0$$

$$\Rightarrow \qquad 0 = a_1 + 2\,a_2L$$

$$\Rightarrow \qquad a_1 = -2\,a_2L$$

Solving a_0 and a_1 value in equation (1),

$$(1) \Rightarrow \qquad y(x) = -2\,a_2x\,L + a_2x^2$$

$$y(x) = a_2[x^2 - 2\,xL] \qquad \ldots.(2)$$

$$\Rightarrow \qquad \frac{dy}{dx} = a_2(2x - 2\,L)$$

$$\frac{d^2y}{dx^2} = 2a_2$$

We know that, residual,

$$R = A\,E\,\frac{d^2y}{dx^2} + q_0 = 0$$

$$\Rightarrow \qquad A\,E\,(2\,a_2) + q_0 = 0$$

$\Rightarrow$ $$A\,E\,2a_2 = -q_0$$

$\Rightarrow$ $$a_2 = \frac{-q_0}{2\,A\,E}$$

Substitute a_2 value in equation (2),

$\Rightarrow$ $$y(x) = \frac{-q_0}{2\,A\,E}[x^2 - 2\,x\,L]$$

$$y(x) = \frac{q_0}{2\,A\,E}[2\,x\,L - x^2]$$

Result: Final solution,

$$y(x) = \frac{q_0}{2\,A\,E}[2\,x\,L - x^2]$$

Example 1.9

The governing differential equation for the fully developed lamina flow is given by $\mu\frac{d^2y}{dx^2} + \rho\,g\,cos\,\theta\; = 0.$

if boundary conditions are $\left.\frac{du}{dx}\right|_{X=L} = 0, u(L) = 0,$

Find the velocity distribution, u(x).

Solution: Differential equation,

$$\mu\frac{d^2y}{dx^2} + \rho\,g\,cos\,\theta\; = 0$$

Boundary conditions are $\left.\frac{du}{dx}\right|_{X=0} = 0$

$$u(L) = 0$$

To find: Velocity distribution, u(x).

Solution: Assume a trial function.

Let $$u(x) = a_0 + a_1x + a_2x^2 \ldots \qquad \ldots(1)$$

Apply first boundary condition,

i.e., $$\frac{du}{dx} = 0\;at\;x = 0.$$

$\Rightarrow$ $$\frac{du}{dx} = a_1 + 2\, a_2\, x$$

$AT\ x = 0\,,$ $$\frac{du}{dx} = 0$$

$\Rightarrow$ $$a_1 = 0$$

Apply second boundary condition,

i.*e.,* at *x* = L, *u(x)* = 0

$\Rightarrow$ $$u(x) = \ a_0 + a_1 x + a_2 x^2$$

$\Rightarrow$ $$0 = \ a_0 + a_1 L + a_2 L^2$$

Substitute $a_1 = 0$

$\Rightarrow$ $$0 = \ a_0 + a_2 L^2$$

$\Rightarrow$ $$a_0 = -a_2 L^2$$

Solving $a_0\ and\ a_1$ value in equation (1),

$$u(x) = \ -a_2 L^2 + 0 + a_2 x^2$$

$$= a_2[x^2 - L^2]$$

$$u(x) = a_2[x^2 - L^2] \qquad \text{... (2)}$$

$\Rightarrow$ $$\frac{du}{dx} = a_2[2\, x]$$

$$\frac{d^2u}{dx^2} = 2\, a_2$$

We know that, residual,

$$R = \mu \frac{d^2y}{dx^2} + \rho\ g\ cos\ \theta = 0$$

$\Rightarrow$ $$\mu\,(2\, a_2) + \rho\ g\ cos\ \theta = 0$$

$$a_2 = \frac{-\ \rho\ g\ cos\theta}{2\ \mu}$$

Substitute a_2 value in equation (2),

$$\Rightarrow \qquad u(x) = \frac{-\rho\, g\, cos\theta}{2\,\mu}[x^2 - L^2]$$

$$u(x) = \frac{\rho\, g\, cos\theta}{2\,\mu}[L^2 - x^2]$$

Result: Velocity distribution

$$u(x) = \frac{\rho\, g\, cos\theta}{2\,\mu}[L^2 - x^2]$$

Example 1.10

Find the solution for the following differential equation.

$$E\,I\frac{d^4u}{dx_4} - q_0 = 0$$

The boundary conditions are u(0) = 0, $\frac{du}{dx}(0) = 0$

$$\frac{d^2u}{dx^2}(L) = 0, \qquad \frac{d^3u}{dx^3}(L) = 0$$

Given: The governing differential equation

$$E\,I\frac{d^4u}{dx_4} - q_0 = 0$$

The boundary conditions are u(0) = 0, $\frac{du}{dx}(0) = 0$

$$\frac{d^2u}{dx^2}(L) = 0, \qquad \frac{d^3u}{dx^3}(L) = 0$$

Solution: Assume a trial function.

$$Let \qquad u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots \qquad \ldots(1)$$

Apply first boundary condition,

i.e., at $x = 0$, $u(x) = 0$

$$\Rightarrow \qquad 0 = a_0 + 0 + 0 + 0 + 0$$

$$\Rightarrow \qquad a_0 = 0$$

Apply second boundary condition,

i.e., at $x = 0,\qquad \dfrac{du}{dx} = 0$

$$\Rightarrow \qquad \frac{du}{dx} = 0 + \mathrm{a}_1 + 2\,a_2 + 3\,a_3x^2 + 4\,a_4x^3$$

$$0 = a_1 + 0 + 0 + 0$$

$$\Rightarrow \qquad a_1 = 0$$

Apply third boundary condition,

i.e., at $\mathrm{x} = \mathrm{L},\qquad \dfrac{d^2u}{dx^2} = 0$

$$\Rightarrow \qquad \frac{d^2u}{dx^2} = 2\,\mathrm{a}_2 + 6\,a_3 + 12\,a_4x^2$$

$$\Rightarrow \qquad 0 = 2\,a_2 + 6\,a_3L + 12\,a_4\,L^2$$

$$\Rightarrow \qquad 2\,a_2 = -6\,a_3L - 12\,a_4\,L^2$$

$$\Rightarrow \qquad a_2 = -[3\,a_3L + 6\,a_4L^2]$$

Apply fourth boundary condition,

i.e., at $\qquad x = \mathrm{L}, \dfrac{d^3u}{dx^3} = 0$

$$\Rightarrow \qquad \frac{d^3u}{dx^3} = 0 + 6\,a_3 + 24\,a_4x$$

$$\Rightarrow \qquad 0 = 6\,a_3 + 24\,a_4\,L$$

$$\Rightarrow \qquad 6\,a_3 = -24\,a_4\,L$$

$$\Rightarrow \qquad a_3 = -4\,a_4L$$

Substitute a_0, a_1, a_2 and a_3 values in equation (1),

$$u(x\,) = 0 + 0 - [3\,a_3L + 6\,a_4L^2]x^2 - 4\,a_4L\,x^3 + a_4x^4$$

$$= -[3\, a_3 L + 6\, a_4 L^2]x^2 - 4\, a_4 L\, x^3 + a_4 x^4$$

$$= -[3\,(-4\, a_4 L) \times L\ + 6\, a_4 L^2]x^2 - 4\, a_4 L\, x^3 + a_4 x^4$$

$$[\because a_3 = -4\, a_4 L]$$

$$= 12\, a_4 L^2 x^2 - 6\, a_4 L^2 x^2 - 4\, a_4 L\, x^3 + a_4 x^4$$

$$= a_4[12\, L^2 x^2 - 6\, L^2\, x^2 - 4\, L\, x^3 + x^4]$$

$$u(x) = a_4[6\, L^2 x^2 - 4\, L\, x^3 + x^4] \qquad \ldots (2)$$

$$\Rightarrow \quad \frac{du}{dx} = a_4[6\, L^2(2x) - 12\, L\, x^2 + 4\, x^3]$$

$$\Rightarrow \quad \frac{d^2u}{dx^2} = a_4[6\, L^2(2) - 24\, L\, x + 12\, x^2]$$

$$\Rightarrow \quad \frac{d^3u}{dx^3} = a_4[0 - 24L + 24x]$$

$$\Rightarrow \quad \frac{d^4u}{dx^4} = a_4[0 - 0 + 24]$$

$$\frac{d^4u}{dx^4} = 24\, a_4$$

We know that, Residual,

$$R = E\, I \frac{d^4u}{dx_4} - q_0 = 0$$

$$\Rightarrow \quad E\, I\, (24\, a_4) - q_0 = 0$$

$$\Rightarrow \quad E\, I\, 24\, a_4 = q_0$$

$$\Rightarrow \quad a_4 = \frac{q_0}{24\, E\, I}$$

Substitute a_4 value in equation (2),

$$\Rightarrow \quad u(x) = \frac{q_0}{24\, E\, I}[\, 6\, L^2 x^2 - 4\, L\, x^3 + x^4]$$

$\Rightarrow$ $$u(x) = \frac{q_0}{24\,E\,I}\left[\,x^4 - 4\,L\,x^3 + 6\,L^2x^2\right]$$

Result: Final solution

$$u(x) = \frac{q_0}{24\,E\,I}\left[\,x^4 - 4\,L\,x^3 + 6\,L^2x^2\right]$$

Example 1.11

The following differential equation is available for a physical phenomenon.

$$A\,E\,\frac{d^2u}{dx^2} + a\,x = 0$$

The boundary conditions are $u(0) = 0, A\,E\,\left.\frac{du}{dx}\right|_{X=L} = 0$. By using Galerkin's technique, find the solution of the above differential equation.

Given: Differential equation,

$$A\,E\,\frac{d^2u}{dx^2} + a\,x = 0$$

Boundary condition $u(0) = 0,\; A\,E\,\left.\frac{du}{dx}\right|_{X=L} = 0$

To find: u(x) by using Galerkin's technique.

Solution: Assume a trial function.

Let $$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \ldots \qquad \ldots(1)$$

Apply first boundary condition, i.e., at $x = 0$, $u(x) = 0$

$(1) \Rightarrow$ $$0 = a_0 + 0 + 0 + 0$$

$\Rightarrow$ $$a_0 = 0$$

Apply second boundary condition,

i.e., at $$x = \mathrm{L}, A\,E\,\frac{du}{dx} = 0$$

$\Rightarrow$ $$\frac{du}{dx} = 0 + \mathrm{a}_1 + 2\,a_2x + 3\,a_3x^2$$

$\Rightarrow$ $$0 = \mathrm{a}_1 + 2\,a_2L + 3\,a_3L^2$$

$\Rightarrow \qquad a_1 = -(2\, a_2 L + 3\, a_3 L^2)$

Substitute a_0 *and* a_1 value in equation (1),

$$u(x\,) = 0 + -(2\, a_2 L + 3\, a_3 L^2)x - a_2\, x^2 + a_3 x^3$$

$$= -2\, a_2 Lx + 3\, a_3 L^2 x + \ a_2\, x^2 + a_3 x^3$$

$$= \ a_2[x^2 - 2\, Lx] + \ a_3[x^3 - 3\, L^2 x]$$

$$u(x) = \ a_2[x^2 - 2\, Lx] + \ a_3[x^3 - 3\, L^2 x] \qquad \text{... (2)}$$

We know that, Residual,

$$R = \ A\, E \frac{d^2 u}{dx^2} + a\, x \qquad \text{.. (3)}$$

(2) $\Rightarrow$

$$\frac{du}{dx} = \ a_2[2\, x - 2L] + a_3[3x^2 - 3\, L^2]$$

$$\frac{d^2 u}{dx^2} = \ a_2[2] + a_3[6\, x]$$

$$\frac{d^2 u}{dx^2} = 2\, a_2 + 6\, a_3 x$$

Substitute $\frac{d^2 u}{dx^2}$ value in equation (3),

(3)$\Rightarrow \qquad R = A\, E\, (2\, a_2 + 6\, a_3\, x) + a\, x$

Residual, $\qquad R = A\, E\, (2\, a_2 + 6\, a_3\, x) + a\, x \qquad$ ... (4)

From Galerkin's technique,

$$\int_0^L w_i R\, dx = 0 \qquad \text{... (5)}$$

where, $\mathrm{w_i}$ = weighting function

From equation (2), we know that,

$$w_1 = (x^2 - 2\, L\, x)$$

$$w_2 = (x^3 - 3\, L^2\, x)$$

Substitute w_1, w_2 and R values in equation (5),

$$(5) \Rightarrow \int_0^L (x^2 - 2\,L\,x)[\,A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x]dx = 0 \qquad \ldots (6)$$

$$\int_0^L (x^3 - 3\,L^2\,x)[\,A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x]dx = 0 \qquad \ldots (7)$$

$$(6) \Rightarrow \int_0^L (x^2 - 2\,L\,x)[\,A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x]dx = 0$$

$$\int_0^L (x^3 - 2\,L\,x)[\,2\,a_2 A\,E + 6\,a_3 + A\,E\,x + a\,x\,]dx = 0$$

$$\int_0^L [2\,a_2 A\,E\,x^2 + 6\,a_3\,A\,E\,x^3 + a\,x^3 - 4\,a_2\,A\,E\,L\,x - 12\,a_3\,A\,E\,L\,x^2$$
$$- 2\,a\,L\,x^2]\,dx = \;0$$

$$\Rightarrow \left[2\,a_2 A\,E\frac{x^3}{3} + 6\,a_3\,A\,E\frac{x^4}{4} + a\,\frac{x^4}{4} - 4\,a_2\,A\,E\,L\frac{x^2}{x} - 12\,a_3\,A\,E\,L\,\frac{x^3}{3}\right.$$
$$\left. - 2\,a\,L\,\frac{x^3}{3}\right]_0^L = 0$$

$$\Rightarrow 2\,a_2 A\,E\frac{L^3}{3} + 6\,a_3\,A\,E\frac{L^4}{4} + a\,\frac{L^4}{4} - 4\,a_2\,A\,E\,\frac{L^3}{x} - 12\,a_3\,A\,E\,\frac{L^4}{3} - 2\,a\,\frac{L^4}{3} = 0$$

$$\Rightarrow \frac{2}{3}\,a_2 A\,E\,L^3 + \frac{3}{2} a_3\,A\,E L^4 + \frac{a\,L^4}{4} - 2\,a_2\,A\,E\,L^3 - 4\,a_3\,A\,E\;L^4 - \frac{2}{3}\,a\,L^4 = 0$$

$$\Rightarrow A\,E a_2\,L^3\left[\frac{2}{3} - 2\right] + a_3\,A\,E L^4\left[\frac{3}{2} - 4\right] + \frac{a\,L^4}{4} - \frac{2}{3} a L^4 = 0$$

$$\Rightarrow \frac{-4}{3} A\,E\,L^3 a_2 - \frac{5}{2}\,A\,E\,L^4\,a_3 = \left[\frac{2}{3} - \frac{1}{4}\right] a\,L^4$$

$$\Rightarrow \frac{-4}{3} A\,E\,L^3 a_2 - \frac{5}{2}\,A\,E\,L^4\,a_3 = \frac{5}{12} a\,L^4$$

$$\Rightarrow \frac{4}{3} A\ E\ a_2\ L^3 + \frac{5}{2}\ A\ E\ \ a_3 L^4 = \ \frac{-5}{12} a\ L^4 \qquad \text{... (8)}$$

Equation (7),

$$\Rightarrow \int_0^L (x^2 - 3\ L^2\ x)[\ A\ E\ (2\ a_2 + 6\ a_3 x) + a\ x] dx = 0$$

$$\Rightarrow \int_0^L [x^3 - 3\ L^2\ x][\ 2\ a_2\ A\ E + 6\ a_3\ A\ E\ x + a\ x\ ] dx = 0$$

$$\Rightarrow \int_0^L [\ 2\ A\ E\ a_2\ x^3 + 6\ A\ E\ a_3\ x^4 + a\ x^4 - 6\ A\ E\ a_2 L^2\ x - 18\ A\ E\ a_3 L^2 x^2$$
$$- 3\ \ a\ L^2 x^2\ ] dx = 0$$

$$\Rightarrow \left[2\ A\ E\ a_2 \frac{x^4}{4} + 6A\ E\ a_3 \frac{x^5}{5} + \frac{a\ x^5}{5} - 6\ A\ E\ a_2 L^2\ \frac{x^2}{2} - 18AE\ a_3 L^2 \frac{x^3}{3} \right.$$
$$\left. - 3\ a\ L^2 \frac{x^3}{x} \right]_0^L = 0$$

$$\Rightarrow \left[\frac{1}{2} A\ E\ a_2\ x^4 \frac{6}{5} + AE\ a_3 x^5 \frac{1}{5} a\ x^5 - 3AE\ a_2 L^2\ x^2 - 6\ AE\ a_3 L^2\ x^3 - a\ L^2 x^3 \right]_0^L$$

$$\Rightarrow \frac{1}{2} AE\ a_2\ L^4 + \frac{6}{5}\ AE\ a_3\ L^5 + \frac{1}{5}\ a\ L^5 - 3\ AE\ a_2 L^2 (L^2) - 6\ AE\ a_3 L^2 (L^3)$$
$$- a\ L^2 (L^3)\ = 0$$

$$\Rightarrow \frac{1}{2} AE\ a_2\ L^4 + \frac{6}{5}\ AE\ a_3\ L^5 + \frac{1}{5}\ a\ L^5 - 3\ AE\ a_2 L^4 - 6\ AE\ a_3 L^5 - a\ L^5\ = 0$$

$$\Rightarrow A\ E a_2\ L^4 \left[\frac{1}{2} - 3\right] + \ AE\ a_3\ L^5 \left[\frac{6}{5} - 6\right] + a\ L^5 \left[\frac{1}{5} - 1\right] = 0$$

$$\Rightarrow A\ E\ L^4 \left[\frac{-5}{2}\right] - \frac{24}{5}\ AE\ a_3\ L^5\ = \frac{4}{5} a\ L^5$$

$$\Rightarrow \frac{5}{2} AE\ a_2\ L^4 + \frac{24}{5}\ AE\ \ a_3 L^5 = \frac{-4}{5} a\ L^5 \qquad \ldots (9)$$

Solving equations (8) and (9),

Equation (8)⇒ $$\frac{4}{3} A\ E\ a_2\ L^3 + \frac{5}{2}\ AE\ \ a_3 L^4 = \frac{-5}{12} a\ L^4$$

Equation (9)⇒ $$\frac{5}{2} AE\ a_2\ L^4 + \frac{24}{5}\ AE\ \ a_3 L^5 = \frac{-4}{5} a\ L^5$$

Multiplying equation (8) by $\frac{5}{2}$ L and equation (9) by $\frac{4}{3}$,

$$\frac{20}{6} A\ E\ a_2\ L^4 + \frac{25}{4}\ AE\ \ a_3 L^5 = \frac{-25}{24} a\ L^5$$

$$\frac{20}{6} AE\ a_2\ L^4 + \frac{96}{15}\ AE\ \ a_3 L^5 = \frac{-16}{15} a\ L^5$$

Subtracting, $$\left(\frac{25}{4} - \frac{96}{15}\right) AE\ a_3\ L^5 = \left(\frac{16}{15} - \frac{25}{24}\right) a\ L^5$$

$$\left(\frac{375 - 384}{60}\right) AE\ a_3\ L^5 = \left(\frac{384 - 375}{360}\right) a\ L^5$$

⇒ $$-\frac{9}{60}\ AE\ a_3 L^5 = \frac{9}{360} a\ L^5$$

⇒ $$-\ 0.15\ AE\ a_3 = 0.025\ a$$

⇒ $$a_3 = -0.1666 \frac{a}{A\ E}$$

⇒ $$a_3 = \frac{-a}{6\ A\ E} \qquad \ldots (10)$$

Substituting a_3 value in equation (8),

$$\frac{4}{3} AE\ a_2\ L^3 + \frac{5}{2}\ AE \left(\frac{-a}{6\ AE}\right) L^4 = \frac{-5}{12}\ a\ L^4$$

$$\frac{4}{3} AE\; a_2\; L^3 = \frac{-5}{12}\; a\; L^4 - \frac{5}{2} AE L^4 \left(\frac{-a}{6\; AE}\right)$$

$$\frac{4}{3} AE\; a_2\; L^3 = \frac{-5}{12}\; a\; L^4 + \frac{5}{2} a\; L^4$$

$$\frac{4}{3} AE\; a_2\; L^3 = 0$$

$\Rightarrow$ $$a_2 = 0$$

Substitute a_2 *and* a_3 values in equation (2),

$\Rightarrow$ $$u(x) = 0\; \times [x_2 - 2\; L\; x] + \left(\frac{-a}{6\; AE}\right)[x^3 - 3\; L^2 x]\; = 0$$

$\Rightarrow$ $$u(x) = \frac{a}{6\; AE}\; [\; 3\; L^2 x - x^3]$$

Result: $$u(x) = \frac{a}{6\; AE}\; [\; 3\; L^2 x - x^3]$$

Example 1.12

The following differential equation for the long cylinder of radius R with heat generation q0 is given by

$$\frac{d^2T}{dr^2} + \frac{I}{r}\frac{dT}{dr} + \frac{q_0}{k} = 0$$

The boundary conditions are T(R) = T_w

$$q_0\; \pi\; R^2\; L = (-k)(2\pi\; R\; L)\frac{dT}{dr}\bigg|_{r=R}$$

Find the temperature distribution T as a function of radial location r.

Given: Differential equation

$$\frac{d^2T}{dr^2} + \frac{I}{r}\frac{dT}{dr} + \frac{q_0}{k} = 0$$

Boundary conditions T(R) = T_w

$$q_0\,\pi\,R^2\,L = (-k)(2\pi\,R\,L)\left.\frac{dT}{dr}\right|_{r=R}$$

To find: Temperature distribution, T(r)

Solution: Assume a trial solution.

$$Let \qquad T = a_0 + a_1(r-R) + a_2(r-R^2) \qquad ....(1)$$

Apply first boundary condition, i.e., at r = R, T = T_w

$$(1) \Rightarrow \qquad T_w = a_0 + a_1(R-R) + a_2(R-R)^2$$

$$T_w = a_0$$

$$a_0 = T_w$$

Apply second boundary condition, i.e.,

$$q_0\,\pi\,R^2\,L = (-k)(2\pi\,R\,L)\left.\frac{dT}{dr}\right|_{r=R}$$

$$(1) \Rightarrow \qquad \frac{dT}{dr} = 0 + a_1(1-0) + a_2\,2\,(r-R)(1-0)$$

$$= a_1 + 2\,a_2(r-R)$$

$$\left.\frac{dy}{dx}\right|_{r=R} = a_1 + 2\,a_2(R-R)$$

$$\Rightarrow \qquad \left.\frac{dy}{dx}\right|_{r=R} = a_1$$

We know that,

$$-k\,2\pi\,R\,L\left.\frac{dT}{dr}\right|_{r=R} = q_0\,\pi\,R^2\,L$$

$$\Rightarrow \qquad -\,k\,2\pi\,R\,L(a_1) = q_0\,\pi\,R^2\,L$$

$$\Rightarrow \qquad a_1 = \frac{q_0\,\pi\,R^2\,L}{-k\,2\pi\,R\,L}$$

$$\Rightarrow \qquad a_1 = \frac{-\,q_0\,R}{2\,k}$$

Solving a_0 *and* a_1 value in equation (1),

$$\Rightarrow \qquad T = T_w + \frac{-q_0 R}{2\,k}(r-R) + a_2(r-R)^2$$

$$T = T_w - \frac{q_0 R}{2\,k}(r-R) + a_2(r-R)^2 \qquad \text{... (2)}$$

We know that, Residual,

$$\mathrm{R}\,\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{q_0}{k} \qquad \text{... (3)}$$

$$(2) \Rightarrow \qquad \frac{dT}{dr} = 0 - \frac{q_0\,R}{2\,k} + a_2 2(r-R)$$

$$\frac{d^2T}{dr^2} = 0 + 2a_2(1-0)$$

$$= 2a_2$$

$$(3) \Rightarrow \qquad R = 2\,a_2 + \frac{1}{r}\left(\frac{-q_0\,R}{2\,k} + 2\,a_2\,(r-R)\right) \qquad \text{... (4)}$$

From Galerkin's technique

$$\int_0^R w_i\,R\,dr = 0$$

$$\Rightarrow \qquad 2\pi L\int_0^R (r-R)^2\left[2\,a_2 + \frac{q_0}{k} + \frac{1}{r}\left(\frac{-q_0 R}{2k} + \;2\,a_2(r-R)\right)\right] r\,dr = 0$$

$$\Rightarrow \qquad 2\pi L\int_0^R (r-R)^2\left[2\,a_2 + \frac{q_0}{k} - \frac{q_0 R}{2\,k\,r} + \frac{2\,a_2}{r}(r-R)\;\right] r\,dr = 0$$

$$\Rightarrow \qquad 2\pi L\int_0^R (r-R)^2\left[2\,a_2 r + \frac{q_0\,r}{k} - \frac{q_0 R}{2\,k} + \;2\,a_2\,(r-R)\;\right] dr = 0$$

$$\Rightarrow \qquad 2\pi L \underbrace{\int_0^R (r-R)^2}_{u} \underbrace{\left[2\,a_2 r + \frac{q_0\,r}{k} - \frac{q_0 R}{2\,k} + \ 2\,a_2\,r - 2\,a_2\,R\ \right]}_{v} dr = 0$$

$$\ldots (5)$$

Now apply Bernoull's formula,

$$\left[\int u\,v\,dx = u\,v_1 - u'v_2 + u''v_3 + \ u'''\,v_4 + \cdots\right]$$

u = Differentiate

v = integrate

Differentiating u with respect to r,

$$u = (r-R)^2$$

$$u' = 2\,(r-R)$$

$$u'' = 2$$

$$u''' = 0$$

Integrating v with respect to r,

$$v = 2\,a_2 r + \frac{q_0}{k} r - \frac{q_0}{2\,k} R + 2a_2 r - 2\,a_2 R$$

$$v_1 = 2\,a_2 \frac{r^2}{2} + \frac{q_0}{k}\frac{r^2}{2} - \frac{q_0}{2\,k} R\,r + 2a_2\,\frac{r^2}{2} - 2\,a_2 R\,r$$

$$\Rightarrow \qquad v_2 = \ a_2 \frac{r^3}{3} + \frac{q_0}{2\,k}\frac{r^3}{3} - \frac{q_0}{2\,k} R\,\frac{r^2}{2} + a_2\,\frac{r^3}{3} - 2\,a_2 R\,\frac{r^2}{2}$$

$$v_3 = \frac{a_2}{3}\,\frac{r^4}{3} + \frac{q_0}{6\,k}\frac{r^4}{4} - \frac{q_0}{4\,k} R\,\frac{r^3}{3} + \frac{a_2}{3}\,\frac{r^4}{4} - \ a_2 R\,\frac{r^3}{3}$$

$$= \frac{a_2\,r^4}{12} + \frac{q_0\,r^4}{24\,k} - \frac{q_0\,R\,r^3}{12\,k} + \frac{a_2\,r^4}{12} - \ \frac{a_2\,R\,r^3}{3}$$

Now substitute the above values in the Bernoulli's formula

$$\int u\,v\,dx = u\,v_1 - u' v_2 + u'' v_3 + u''' v_4 + \cdots$$

$$= (r-R)^2\left[a_2\,r^2 + \frac{q_0 r^2}{2\,k} - \frac{q_0 R\,r}{2\,k} + a_2\,r^2 - 2\,a_2\,r\,R\right]$$

$$= 2(r-R)\left[\frac{a_2\,r^3}{3} + \frac{q_0 r^3}{6\,k} - \frac{q_0 R\,r^2}{4\,k} + \frac{a_2\,r^3}{3} - a_2\,R\,r^2\right] + 2\left[\frac{a_2\,r^4}{12} + \frac{q_0 r^4}{24\,k} - \frac{q_0 R\,r^3}{12\,k} + \frac{a_2\,r^4}{12} - \frac{a_2\,R\,r^2}{3}\right]$$

Substitute in equation (5),

$$2\pi\,L\left[(r-R)^2\left(a_2\,r^2 + \frac{q_0 r^2}{2\,k} - \frac{q_0 R\,r}{2\,k} + a_2\,r^2 - 2\,a_2\,r\,R\right)\right] - 2(r-R)\left[\frac{a_2\,r^3}{3} + \frac{q_0 r^3}{6\,k} - \frac{q_0 R\,r^2}{4\,k} + \frac{a_2\,r^3}{3} - a_2\,R\,r^2\right] + 2\left[\frac{a_2\,r^4}{12} + \frac{q_0 r^4}{24\,k} - \frac{q_0 R\,r^3}{12\,k} + \frac{a_2\,r^4}{12} - a_2\frac{R\,r^2}{3}\right]_0^R = 0$$

$$\Rightarrow \qquad 2\,\pi\,L\left[0 - 0 + 2\left(\frac{a_2\,R^4}{12} + \frac{q_0 R^4}{24\,k} - \frac{q_0 R^4}{12\,k} + \frac{a_2\,R^4}{12} - \frac{a_2\,R^4}{3}\right) - 0\right] = 0$$

$$\Rightarrow \qquad 2\,\pi\,L\left[\frac{a_2\,R^4}{6} + \frac{q_0 R^4}{12\,k} - \frac{q_0 R^4}{6\,k} + \frac{a_2\,R^4}{6} - \frac{2\,a_2\,R^4}{3}\right] = 0$$

$$\Rightarrow \qquad 2\,\pi\,L\left[\frac{2\,a_2\,R^4}{6} + \frac{q_0 R^4}{12\,k} - \frac{q_0 R^4}{6\,k} - \frac{2\,a_2\,R^4}{3}\right] = 0$$

$$\Rightarrow \qquad \left[\frac{a_2\,R^4}{3} - \frac{q_0 R^4}{12\,k} - \frac{2\,a_2\,R^4}{3}\right] = 0$$

$$\frac{-\,a_2\,R^4}{3} = \frac{q_0 R^4}{12\,k}$$

$\Rightarrow$ $$a_2 = \frac{-3\, q_0 R^4}{R^4 \times 12\, k}$$

$$a_2 = \frac{-q_0}{4\, k}$$

Substitute a_2 value in equation (2),

(2)$\Rightarrow$ $$T = T_w + \frac{-q_0 R}{2\, k}(r - R) + \frac{-q_0}{4\, k}(r - R)^2$$

$$= T_w - \frac{q_0 R\, r}{2\, k} + \frac{q_0 R^2}{2\, k} - \frac{q_0}{2\, k}[\, r^2 + R^2 - 2\, r\, R]$$

$$= T_w - \frac{q_0 R\, r}{2\, k} + \frac{q_0 R^2}{2\, k} - \frac{q_0\, r^2}{4\, k} - \frac{q_0\, R^2}{4\, k} + \frac{q_0\, 2\, r\, R}{4\, k}$$

$$= T_w - \frac{q_0 R\, r}{2\, k} + \frac{q_0 R^2}{2\, k} - \frac{q_0\, r^2}{4\, k} - \frac{q_0\, R^2}{4\, k} + \frac{q_0\;\, r\, R}{4\, k}$$

$$= T_w + \frac{q_0 R^2}{k}\left[\frac{1}{2} - \frac{1}{4}\right] - \frac{q_0\, r^2}{4\, k}$$

$$= T_w + \frac{q_0 R^2}{k}\left[\frac{2}{8}\right] - \frac{q_0\, r^2}{4\, k}$$

$$= T_w + \frac{q_0 R^2}{4\, k} - \frac{q_0\, r^2}{4\, k}$$

$$T\ = T_w + \frac{q_0}{4\, k}[R^2 - r^2]$$

$\Rightarrow$ $$T - T_w = \frac{q_0}{4\, k}[R^2 - r^2]$$

Example 1.13

The following differential equation is available for a physical phenomenon.

$$\frac{d^2 y}{dx^2} + 50 = 0, 0\ \le x\ \le 10$$

Trial function is, $y = a_1(10 - x)$

Boundary conditions are, y(0) = 0

y(10) = 0

Find the value of the parameter a_1 by the following methods:

(i) Point collocation; (ii) Subdomain collocation; (iii) Least squares; (iv) Galerkin.

Given:

Differential equation,

$$\frac{d^2y}{dx^2} + 50 = 0, 0 \le x \le 10 \qquad \text{... (1)}$$

Trial function is, $y = a_1 x(10 - x)$

Boundary conditions are, y(0) = 0

y(10) = 0

To find: The value of the parameter a_1 by,

(i) Point collocation method; (ii) Subdomain collocation method;

(iii) Least squares method; (iv) Galerkin's method.

Solution:

First we have to verify, whether the trial function satisfies the boundary conditions or not.

Trial function is, $y = a_1 \, x \, (10 - x)$

When x = 0, y = 0

x = 10, y = 0

Hence it satisfies the boundary conditions.

(i) Point Collocation method: $y = a_1 \, x \, (10 - x)$

$$\Rightarrow \quad y = a_1 \; (10x - x^2)$$

$$\Rightarrow \frac{dy}{dx} = a_1 \; (10 - 2x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\,2 \, a_1$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation (1),

$\Rightarrow$ Residual, $R = -2\,a_1 + 50$ ... (2)

In point collocation method, residuals are set to zero.

$$\Rightarrow \quad R = -2\,a_1 + 50 = 0$$

$$-2\,a_1 = -50$$

$$a_1 = 25 \quad \text{... (3)}$$

Hence the trial function is, *y = 25 x (10- x)*

(ii) Subdomain collocation method:

This method requires, $\int_0^{10} R\,dx = 0$

Substitue R value, $\Rightarrow \int_0^{10} [-2\,a_1 + 50]\,dx = 0$

$$\Rightarrow \int_0^{10} [-2\,a_1 dx + 50\,dx] = 0$$

$$\Rightarrow [-2\,a_1 x + 50\,x]_0^{10} = 0$$

$$\Rightarrow -20\,a_1 = -500$$

$$a_1 = 25 \quad \text{... (4)}$$

Hence the trial function is, y = 25 x (10- x)

(iii) Least squares method:

This method requires $I = \int_0^{10} R^2 dx$

It can also be written as, $\frac{\partial I}{\partial a_1} = \int_0^{10} R\,\frac{\partial I}{\partial a_1} dx$ .. (5)

We know that, $R = -2\,a_1 + 50$

$$\frac{\partial I}{\partial a_1} = -2$$

Substitue R and $\frac{\partial I}{\partial a_1}$ value in equation (5),

$$\Rightarrow \qquad \frac{\partial I}{\partial a_1} \int_0^{10} (-2\,a_1 + 50)(-2)\,dx$$

The requirement is, $\frac{\partial I}{\partial a_1} = 0$

$$\Rightarrow \qquad \int_0^{10} (-2\,a_1 + 50)(-2)\,dx = 0$$

$$\Rightarrow \qquad \int_0^{10} (-2\,a_1 + 50)dx = 0$$

$$\Rightarrow \qquad \int_0^{10} [-2\,a_1\,dx \;+ 50\,dx\,] = 0$$

$$\Rightarrow \qquad [-2\,a_1 + 50\,x]_0^{10} = 0$$

$$\Rightarrow \qquad -\,2\,\mathrm{a}_1(10) + 50(10) - [0] = 0$$

$$\Rightarrow \qquad -\,2\,\mathrm{a}_1 + 500 = 0$$

$$\Rightarrow \qquad -\,2\,\mathrm{a}_1 = -500$$

$$\Rightarrow \qquad \mathrm{a}_1 = 25 \qquad \ldots(6)$$

Therefore, the trial; function becomes, y = 25 x (10 –x).

(iv) Galerkin's method:

In this method, the trial function itself is considered as the weighting function , w_i

$$\Rightarrow \int_0^{10} w_i \, R \, dx = 0 \qquad \dots(7)$$

Here, the trial function is, $y = w_i = a_1 \, x(10 - x)$

Substitute w_i *and* R Values in equation (7),

$$\Rightarrow \int_0^{10} a_1 x(10 - x) \times (-2\, a_1 + 50)\, dx = 0$$

$$\Rightarrow a_1 \int_0^{10} x(10 - x) \times (-2\, a_1 + 50)\, dx = 0$$

$$\Rightarrow a_1 \int_0^{10} (10x - x^2)(-2\, a_1 + 50)\, dx = 0$$

$$\Rightarrow a_1 \int_0^{10} [-20\, a_1 x + 500\, x + 2\, a_1\, x^2 - 50\, x^2] dx = 0$$

$$\Rightarrow a_1 \left[-20\, a_1 \frac{x^2}{2} + 500 \frac{x^2}{2} + 2\, a_1 \frac{x^3}{3} - 50 \frac{x^3}{3} \right]_0^{10} = 0$$

$$\Rightarrow \frac{-20\, a_1}{2}[10^2 - 0] + \frac{500}{2}[10^2 - 0] + \frac{2\, a_1}{3}[10^3 - 0] - \frac{50}{3}[10^3 - 0] = 0$$

$$\Rightarrow - 10\mathrm{a}_1[100] + 250\,[100] + \frac{2a_1}{3}\,[\,1000] - \frac{50}{3}\,[\,1000] = 0$$

$$\Rightarrow - 1000\mathrm{a}_1 + 25000 + 666.66\, a_1 - 16666.66 = 0$$

$$\Rightarrow - 333.33\, \mathrm{a}_1 = -8333.33$$

$$\Rightarrow \mathrm{a}_1 = 25 \qquad \dots(8)$$

The trial function is, $y = 25\, x\, (10 - x)$

From equation (3), (4), (6) and (8) we know that the value of parameter a_1 is same for all the four methods.

Result: Parameter, a_1 (for all the four methods) = 25

Example 1.14

The differential equation of a physical phenomenon is given by,

$$\frac{d^2y}{dx^2} + 500\,x^2 = 0, 0 \le x \le 1$$

Trial function is, $y = a_1(x - x^4)$

Boundary conditions are, y(0) = 0

y(1) = 0

calculate the value of the parameter a_1 by the following methods:

(i) Point collocation; (ii) Subdomain collocation; (iii) Least squares; (iv) Galerkin.

Given:

Differential equation $\frac{d^2y}{dx^2} + 500\,x^2 = 0, 0 \le x \le 1$ ... (1)

Trial function is, $y = a_1(x - x^4)$

Boundary conditions are, y(0) = 0

y(1) = 0

To find: The value of the parameter a_1 by,

(i) Point collocation method;

(ii) Subdomain collocation method;

(iii) Least squares method;

(iv) Galerkin's method.

Solution:

First, we have to verify whether the trial function satisfies the boundary conditions or not.

Trial function is, $y = a_1\ (x - x^4)$

When x = 0, y = 0

x = 1, y = 0

Hence it satisfies the boundary conditions.

(i) Point Collocation method: $y = a_1\ (x - x^4)$

$$\Rightarrow \frac{dy}{dx} = a_1\ (1 - 4x^3)$$

$$\Rightarrow \frac{d^2y}{dx^2} = a_1\ (0 - 12x^2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\ 12\ a_1 x^2$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation (1),

$\Rightarrow$ Residual, $\qquad R = -12\ a_1 x^2 + 500\ x^2 \qquad$ ... (2)

In point collocation method, residuals are set to zero.

$\Rightarrow \qquad R = -12\ a_1 x^2 + 500\ x^2 = 0 \qquad$ ... (3)

In this problem, we have to find only one parameter, a_1. So, only one collocation point is needed. The point may be chosen between 0 and 1. Let us take $\frac{1}{2}$.

Substituting $x = \frac{1}{2}$ in equation (3)

$$\Rightarrow \qquad R = -12\ a_1 \left(\frac{1}{2}\right)^2 + 500\ \left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow \qquad -\ 12\ a_1 \left(\frac{1}{4}\right) + 500 \left(\frac{1}{4}\right) = 0$$

$$\Rightarrow \qquad -\ 3\ a_1 + 125 = 0$$

$$\Rightarrow \qquad a_1 = 41.66 \qquad \text{... (4)}$$

Hence the trial function is, $y = 41.66\ (x - x^4)$

(ii) Subdomain collocation method:

This method requires, $\int_0^1 R\,dx = 0$

Substitue R value,

$$\Rightarrow \quad \int_0^1 (-12\,a_1 x^2 + 500\,x^2)\,dx = 0$$

$$\Rightarrow \quad -12\,a_1 \left[\frac{x^3}{3}\right]_0^1 + 500\left[\frac{x^3}{3}\right]_0^1 = 0$$

$$\Rightarrow \quad \frac{-12\,a_1}{3}[1-0] + \frac{500}{3}[1-0] = 0$$

$$\Rightarrow \quad \frac{-12\,a_1}{3} + \frac{500}{3} = 0$$

$$\Rightarrow \quad -12\,a_1 + 500 = 0$$

$$\Rightarrow \quad -12\,a_1 = -500$$

$$\Rightarrow \quad a_1 = \frac{500}{12}$$

$$\Rightarrow \quad a_1 = 41.66 \qquad \text{...(5)}$$

Trial function is, $y = 41.66\,(x - x^4)$

(iii) Least squares method:

This method requires $\quad I = \int_0^1 R^2 dx$

It can also be written as, $\frac{\partial I}{\partial a_1} = \int_0^1 R\,\frac{\partial R}{\partial a_1}dx \qquad ..(6)$

We know that, $\quad R = -12\,a_1 x^2 + 500\,x^2$

$$\frac{\partial R}{\partial a_1} = -12x^2$$

Substitue R and $\frac{\partial R}{\partial a_1}$ value in equation (6),

$$\Rightarrow \qquad \frac{\partial I}{\partial a_1} = \int_0^1 (-12\, a_1 x^2 + 500\, x^2)(-12\, x^2)\, dx$$

The requirement is, $\frac{\partial I}{\partial a_1} = 0$

$$\Rightarrow \quad \int_0^1 (-12\, a_1 x^2 + 500\, x^2)(-12\, x^2)\, dx = 0$$

$$\Rightarrow \quad \int_0^1 (144\, a_1 x^4 - 6000\, x^4) dx = 0$$

$$\Rightarrow \quad \int_0^1 (144\, a_1 x^4 - 6000\, x^4) dx = 0$$

$$\Rightarrow \quad 144\, a_1 \left[\frac{x^5}{5}\right]_0^1 + 6000\, \left[\frac{x^5}{5}\right]_0^1 = 0$$

$$\Rightarrow \quad \frac{144\, a_1}{5} [1 - 0] + \frac{6000}{5}\, [1 - 0] = 0$$

$$\Rightarrow \quad 28.8\, \mathrm{a}_1 = 1200$$

$$\Rightarrow \quad \mathrm{a}_1 = 41.66 \qquad \ldots (7)$$

The trial function is, $y = 41.66\ (x - x^4)$.

(iv) Galerkin's method: In this method, the trial function itself is considered as the weighting function , w_i

$$\Rightarrow \qquad \int_0^1 w_i\, R\, dx = 0 \qquad \ldots (8)$$

Here, the trial function is, $\qquad y = w_i = a_1\, (x - x^4)$

Substitute w_i and R Values in equation (8),

$$\Rightarrow \qquad \int_0^1 a_1(x - x^4)(-12\,a_1x^2 + 500\,x^2)\,dx = 0$$

$$\Rightarrow \qquad a_1\int_0^1 (x - x^4)(-12\,a_1x^2 + 500\,x^2)\,dx = 0$$

$$\Rightarrow \qquad a_1\int_0^1 (-12\,a_1x^3 + 500\,x^3 + 12\,a_1x^6 + 500\,x^6\,)\,dx = 0$$

$$\Rightarrow \quad a_1\left[-12\,a_1\left(\frac{x^4}{4}\right)_0^1 + 500\left(\frac{x^4}{4}\right)_0^1 + 12\,a_1\left(\frac{x^7}{7}\right)_0^1 - 500\left(\frac{x^7}{7}\right)_0^1\right] = 0$$

$$\Rightarrow \quad \frac{-12\,a_1}{4}[1-0] + \frac{500}{4}[1-0] + \frac{12\,a_1}{7}[1-0] - \frac{500}{7}[1-0] = 0$$

$$\Rightarrow \qquad -3\mathrm{a}_1 + 125 + 1.714\,a_1 - 71.428 = 0$$

$$\Rightarrow \qquad -1.286\,\mathrm{a}_1 = -53.572$$

$$\Rightarrow \qquad \mathrm{a}_1 = 41.66 \qquad \text{... (9)}$$

Trial function is, $y = 41.66\ (x - x^4)$

From equation (4), (5), (7) and (9) we know that the value of parameter a_1 is same for all the four methods.

Result: Parameter, a_1 (For all the four methods) = 41.66.

Example 1.15

The differential equation of a physical phenomenon is given by,

$$\frac{d^2y}{dx^2} + 500\,x^2 = 0;\ 0\ \le x\ \le 1$$

By using the trial function, $y = a_1(x - x^3) + a_2(x - x^5)$, calculate the value of the parameter a_1 and a_2 by the following methods:

(i) Point collocation; (ii) Subdomain collocation; (iii) Least squares; (iv) Galerkin.

The boundary conditions are: y(0) = 0

y(1) = 0

Given:

Differential equation $\frac{d^2y}{dx^2} + 500\,x^2 = 0, 0\ \leq x\ \leq 1$ ... (1)

Trial function is, $y = a_1(x - x^3) + a_2(x - x^5)$

Boundary conditions are: y(0) = 0

y(1) = 0

To find: The value of the parameter $a_1\ and\ a_2$ by,

(i) Point collocation method;

(ii) Subdomain collocation method;

(iii) Least squares method;

(iv) Galerkin's method.

Solution:

First we have to verify, whether the trial function satisfies the boundary conditions or not.

Trial function is $y = a_1(x - x^3) + a_2(x - x^5)$

When x = 0, y = 0

x = 1, y = 0

Hence it satisfies the boundary conditions.

Residual, R: $y = a_1(x - x^3) + a_2(x - x^5)$

$$\frac{dy}{dx} = a_1(1 - 3\,x^2) + a_2(1 - 5\,x^4)$$

$$\frac{d^2y}{dx^2} = a_1(-6x) + a_2(-20\,x^3)$$

$$\frac{d^2y}{dx^2} = -6\, a_1 x - 20\, a_2 x^3$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation (1),

Residual, $\quad R = -6\, a_1 x - 20\, a_2 x^3 + 500\, x^2 \quad \ldots (2)$

The interval 0 to 1 is divided into two domains $0\ to \frac{1}{2}\ and \frac{1}{2}\ to\ 1$.

(i) Point Collocation method: In point collocation method, residuals are set to zero.

$$\Rightarrow \quad R = -6\, a_1 x - 20\, a_2 x^3 + 500\, x^2 = 0 \quad \ldots (3)$$

Domain (1): Limits is 0 to $\frac{1}{2}$: In domain (1), we can choose an arbitrary point. Let it be $\frac{1}{3}$.

So, put $x = \frac{1}{2}$ in equation (3).

$$\Rightarrow \quad R = -6\, a_1 \left(\frac{1}{3}\right) - 20\, a_2 \left(\frac{1}{3}\right)^3 + 500\, \left(\frac{1}{3}\right)^2 = 0$$

$$\Rightarrow \quad -2\, a_1 - \frac{20}{27}\, a_2 + \frac{500}{9} = 0$$

$$\Rightarrow \quad -2\, a_1 - 0.741\, a_2 = -55.55$$

$$\Rightarrow \quad 2\, a_1 - 0.741\, a_2 = 55.55$$

$$\Rightarrow \quad a_1 + 0.3705\, a_2 = 27.775 \quad \ldots (4)$$

Domain (2) : Limits is $\frac{1}{2}$ *to* 1: In domain (2), we can choose $x = \frac{2}{3}$ and substituting the same in equation (3).

$$\Rightarrow \quad R = -6\, a_1 \left(\frac{2}{3}\right) - 20\, a_2 \left(\frac{2}{3}\right)^3 + 500\, \left(\frac{2}{3}\right)^2 = 0$$

$$\Rightarrow \quad -4\, a_1 - 20\, a_2 \times \frac{8}{27} + 500 \times \frac{4}{9} = 0$$

$$\Rightarrow \quad -4\, a_1 - 5.925\, a_2 + 222.22 = 0$$

$\Rightarrow$ $$4\,a_1 + 5.925\ a_2 = 222.22$$

$\Rightarrow$ $$a_1 + 1.481\,a_2 = 55.555 \qquad \ldots (5)$$

Solving equation (4) and (5)

$$-\,a_1 - 0.3705\,a_2 = -27.775$$
$$a_1 + 1.481\,a_2 = 55.55$$
$$1.111\,a_2 = 55.55$$
$$a_2 = 25$$

Substituting a_2 value in equation (4) or (5).

(5) $\Rightarrow$ $$\mathrm{a}_1 + 1.481(25) = 55.555$$
$$\mathrm{a}_1 + 37.025 = 55.555$$

$\Rightarrow$ $$\mathrm{a}_1 = 18.53$$

Hence the trial function is, $y = 18.53\,(x - x^3) + 25(x - x^5)$

(ii) Subdomain collocation method:

This method requires, $\int_0^1 R\,dx = 0$

The interval 0 to 1 is divided into two domains 0 $to\,\frac{1}{2}\,and\,\frac{1}{2}\,to$ 1.

For domain (1): $\int_0^{1/2} R\,dx = 0$

Substitue R value,

$\Rightarrow$ $$\int_0^{1/2} (-6\,a_1 x + 20a_2 x^3 + 500\,x^2)\,dx = 0$$

$\Rightarrow$ $$-\,6\,a_1\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} - 20\,a_2\,\left[\frac{x^4}{4}\right]_0^{\frac{1}{2}} + 500\left[\frac{x^3}{3}\right]_0^{\frac{1}{2}} = 0$$

$$\Rightarrow \quad \frac{-6\,a_1}{2}\left[\left(\frac{1}{2}\right)^2 - 0\right] - \frac{20\,a_2}{4}\left[\left(\frac{1}{2}\right)^4 - 0\right] + \frac{500}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right] = 0$$

$$\Rightarrow \quad \frac{-6\,a_1}{8} - \frac{20\,a_2}{64} + \frac{500}{24} = 0$$

$$\Rightarrow \quad -0.75\,a_1 - 0.3125\,a_2 + 20.83 = 0$$

$$\Rightarrow \quad -0.75\,a_1 - 0.3125\,a_2 = 20.83$$

$$\Rightarrow \quad a_1 + 0.4166\,a_2 = 27.773 \quad \ldots (6)$$

For domain (2): $\int_{1/2}^{1} R\,dx = 0$

Substitue R value,

$$\Rightarrow \quad -6\,a_1\left[\frac{x^2}{2}\right]_{1/2}^{1} - 20\,a_2\left[\frac{x^4}{4}\right]_{1/2}^{1} + 500\left[\frac{x^3}{3}\right]_{1/2}^{1} = 0$$

$$\Rightarrow \quad \frac{-6\,a_1}{2}\left[1 - \left(\frac{1}{2}\right)^2\right] - \frac{20\,a_2}{4}\left[1 - \left(\frac{1}{2}\right)^4\right] + \frac{500}{3}\left[1 - \left(\frac{1}{2}\right)^3\right] = 0$$

$$\Rightarrow \quad \frac{-6\,a_1}{2}[0.75] - \frac{20\,a_2}{4}[0.9375] + \frac{500}{3}[0.875] = 0$$

$$\Rightarrow \quad -2.25\,a_1 - 4.6875\,a_2 + 145.83 = 0$$

$$\Rightarrow \quad 2.25\,a_1 + 4.6875\,a_2 = 145.83$$

$$\Rightarrow \quad a_1 + 2.083\,a_2 = 64.813 \quad \ldots (7)$$

Solving equation (6) and (7),

$$-a_1 - 0.4166\,a_2 = -27.773$$

$$a_1 + 2.083\,a_2 = 64.813$$

$$1.6664\,a_2 = 37.04$$

$$a_2 = 22.23$$

Substituting a_2 value in equation (6) or (7).

$$(7) \Rightarrow \quad a_1 + 2.083(22.23) = 64.813$$

$$a_1 + 46.305 = 64.813$$

$$\Rightarrow \quad a_1 = 18.50$$

Hence the trial function is, $y = 18.50\,(x - x^3) + 22.23(x - x^5)$

(iii) Least squares method:

This method requires $\quad I = \int_0^1 R^2 dx$

It can also be written as, $\dfrac{\partial I}{\partial a_1} = \int_0^1 R\,\dfrac{\partial R}{\partial a_1} dx \qquad ..(8)$

For domain (1):

$$\frac{\partial I}{\partial a_1} = \int_0^{1/2} R\,\frac{\partial R}{\partial a_1} dx \qquad ..(9)$$

We know that, $R = -6\,a_1 x - 20\,a_2 x^3 + 500\,x^2$

$$\frac{\partial R}{\partial a_1} = -6\,x$$

Substitue R and $\dfrac{\partial R}{\partial a_1}$ value in equation (9),

$$\Rightarrow \quad \frac{\partial I}{\partial a_1} = \int_0^{1/2} (-6\,a_1 x - 20\,a_2 x^3 + 500\,x^2) \times (-6\,x)\,dx$$

The requirement is, $\quad \dfrac{\partial I}{\partial a_1} = 0$

$$\Rightarrow \quad \int_0^{1/2} (-6\,a_1 x - 20\,a_2 x^3 + 500\,x^2) \times (-6\,x)\,dx = 0$$

$$\Rightarrow \quad \int_0^{1/2} (36\, a_1 x^2 + 120\, a_2 x^4 + 3000\, x^3)\, dx = 0$$

$$\Rightarrow \quad 36\, a_1 \left[\frac{x^3}{3}\right]_0^{1/2} + 120\, a_2 \left[\frac{x^5}{5}\right]_0^{1/2} - 3000 \left(\frac{x^4}{4}\right)_0^{1/2} = 0$$

$$\Rightarrow \quad \frac{36\, a_1}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right] + \frac{120\, a_2}{5}\left[\left(\frac{1}{2}\right)^5 - 0\right] - \frac{3000}{4}\left[\left(\frac{1}{2}\right)^4 - 0\right] = 0$$

$$\Rightarrow \quad 12\mathrm{a}_1 \times \frac{1}{8} + 24\, a_2 \times \frac{1}{32} - 750 \times \left(\frac{1}{16}\right) = 0$$

$$\Rightarrow \quad 1.5\, \mathrm{a}_1 + 0.75\, a_2 = 46.875$$

$$\Rightarrow \quad \mathrm{a}_1 + 0.5\, a_2 = 31.25 \ldots (10)$$

For domain (2):

$$\frac{\partial I}{\partial a_1} = \int_0^{1/2} R\, \frac{\partial R}{\partial a_1} dx \qquad ..(11)$$

We know that, $R = -6\, a_1 x - 20\, a_2 x^3 + 500\, x^2$

$$\frac{\partial R}{\partial a_1} = -20\, x^3$$

Substitue R and $\frac{\partial R}{\partial a_2}$ value in equation (11),

$$\Rightarrow \quad \frac{\partial I}{\partial a_2} \int_{1/2}^{1} (-6\, a_1 x - 20\, a_2 x^3 + 500\, x^2) \times (-20\, x^3)$$

The requirement is, $\frac{\partial I}{\partial a_2} = 0$

$$\Rightarrow \quad \int_{1/2}^{1} (-6\, a_1 x - 20\, a_2 x^3 + 500^2) \times (-20\, x^3) = 0$$

$$\Rightarrow \int_{1/2}^{1} 120\, a_1 x^4 + 400\, a_2 x^6 - 10000\, x^5 = 0$$

$$\Rightarrow 120\, a_1 \left[\frac{x^5}{5}\right]_{1/2}^{1} + 400\, a_2 \left[\frac{x^7}{7}\right]_{1/2}^{1} - 10000 \left(\frac{x^6}{6}\right)_{1/2}^{1} = 0$$

$$\Rightarrow \frac{120\, a_1}{5}\left[(1)^5 - \left(\frac{1}{2}\right)^5\right] + \frac{400\, a_2}{7}\left[(1)^7 - \left(\frac{1}{2}\right)^7\right]$$

$$- \frac{10000}{6}\left[(1)^6 - \left(\frac{1}{2}\right)^6 - 0\right] = 0$$

$$\Rightarrow 24\mathrm{a}_1[1 - 0.03125] + 57.142\ \mathrm{a}_2[1 - 0.00781] - 1666.66[1 - 0.01562] = 0$$

$$\Rightarrow 23.25\ \mathrm{a}_1 + 56.695\, a_2 - 1640.626 = 0$$

$$\Rightarrow \mathrm{a}_1 + 2.438\, a_2 = 70.564 \quad \ldots(12)$$

Solving equation (10) and (12),

$$-a_1 - 0.5\, a_2 = -31.25$$

$$a_1 + 2.438\, a_2 = 70.564$$

$$1.938\, a_2 = 39.314$$

$$a_2 = 20.28$$

Substituting a_2 value in equation (10) or (12).

$$(10) \Rightarrow \quad \mathrm{a}_1 + 0.5(20.28) = 31.25$$

$$\mathrm{a}_1 + 10.14 = 31.25$$

$$\Rightarrow \quad \mathrm{a}_1 = 21.11$$

Hence the trial function is, $y = 21.11\,(x - x^3) + 20.28(x - x^5)$

(iv) Galerkin's method: In this method, the trial function itself is considered as the weighting function , w_i

$$\Rightarrow \qquad \int_0^1 w_i\, R\, dx = 0$$

For domain (1): $\displaystyle\int_0^{1/2} w_i\, R\, dx = 0$ ... (13)

Here, the trial function is, $y = w_i = x - x^3$

Residual, R value is, $\mathrm{R} = -6\,a_1 x - 20\,a_2 x^3 + 500\,x^2$

Substitute w_i *and* R Values in equation (13),

$$\Rightarrow \int_0^{1/2} (x - x^3)(-6\,a_1 x + 20\,a_2 x^3 + 500\,x^2)\,dx = 0$$

$$\Rightarrow \int_0^{1/2} [-6\,a_1 x^2 - 20\,a_2 x^4 + 500\,x^3 + 6\,a_1 x^4 + 20\,a_2 x^6 - 500\,x^5]\,dx = 0$$

$$\Rightarrow -6\,a_1\left[\frac{x^3}{3}\right]_0^{1/2} - 20\,a_2\left[\frac{x^5}{5}\right]_0^{\frac{1}{2}} + 500\left[\frac{x^4}{4}\right]_0^{\frac{1}{2}} + 6\,a_1\left[\frac{x^5}{5}\right]_0^{\frac{1}{2}} + 20\,a_2\left[\frac{x^7}{7}\right]_0^{\frac{1}{2}} - 500\left[\frac{x^6}{6}\right]_0^{1/2} = 0$$

$$\Rightarrow -2a_1\left[\left(\frac{1}{2}\right)^3 - 0\right] - 4a_2\left[\left(\frac{1}{2}\right)^5 - 0\right] + 125\left[\left(\frac{1}{2}\right)^4 - 0\right] + 1.2\,a_1\left[\left(\frac{1}{2}\right)^5 - 0\right] + 2.857\,\mathrm{a}_2\left[\left(\frac{1}{2}\right)^7 - 0\right] - 83.33\left[\left(\frac{1}{2}\right)^6 - 0\right] = 0$$

$$\Rightarrow -0.25\,\mathrm{a}_1 - 0.125\,a_1 + 7.8125 + 0.0375\,a_1 + 0.0223\,a_2 - 1.299 = 0$$

$$\Rightarrow -0.2125\mathrm{a}_1 - 0.1027\,a_2 + 6.5135 = 0$$

$$\Rightarrow 0.2125\,\mathrm{a}_1 + 0.1027\,a_2 = 6.5135$$

$$\Rightarrow \ \mathrm{a}_1 + 0.4832\, a_2 = 30.651 \qquad \ldots (14)$$

For domain (2): $\displaystyle\int_{1/2}^{1} w_2\, R\, dx = 0 \qquad \ldots (15)$

Here, the trial function is, $y = w_2 = (x - x^5)$

$$\mathrm{R} = -6\, a_1 x - 20\, a_2 x^3 + 500\, x^2$$

Substitute w_2 *and* R Values in equation (15),

$$\Rightarrow \quad \int_{1/2}^{1} (x - x^5) \times (-6\, a_1 x - 20\, a_2 x^3 + 500\, x^2)\, dx = 0$$

$$\Rightarrow \quad \int_{1/2}^{1} -6\, a_1 x^2 - 20\, a_2 x^4 + 500\, x^3 + 6\, a_1 x^4 + 20\, a_2 x^6 - 500\, x^7 = 0$$

$$\Rightarrow \ -6\, a_1 \left[\frac{x^3}{3}\right]_{1/2}^{1} - 20\, a_2 \left[\frac{x^5}{5}\right]_{1/2}^{1} + 500 \left[\frac{x^4}{4}\right]_{1/2}^{1} + 6\, a_1 \left[\frac{x^7}{5}\right]_{1/2}^{1}$$

$$+ 20\, a_2 \left[\frac{x^9}{9}\right]_{1/2}^{1} - 500 \left[\frac{x^8}{8}\right]_{1/2}^{1} = 0$$

$$\Rightarrow \ -2 a_1 \left[1 - \left(\frac{1}{2}\right)^3\right] - 4\, a_2 \left[1 - \left(\frac{1}{2}\right)^5\right] + 125 \left[1 - \left(\frac{1}{2}\right)^4\right] + 0.857\, a_1 \left[1 - \left(\frac{1}{2}\right)^7\right]$$

$$+ 2.22\, \mathrm{a}_2 \left[1 - \left(\frac{1}{2}\right)^9\right] - 62.5 \left[1 - \left(\frac{1}{2}\right)^8\right] = 0$$

$$\Rightarrow \ -1.75\, \mathrm{a}_1 - 3.875\, a_1 + 117.187 + 0.850\, a_1 + 2.215\, a_2 - 62.225 = 0$$

$$\Rightarrow \ -0.9\, \mathrm{a}_1 - 1.659\, a_2 + 54.932 = 0$$

$$\Rightarrow \ 0.9\, \mathrm{a}_1 + 1.659\, a_2 = 54.932$$

$$\Rightarrow \ \mathrm{a}_1 + 1.843\, a_2 = 61.035 \qquad \ldots (16)$$

Solving equation (14) and (16)

$$-a_1 - 0.4832\, a_2 = -30.651$$

$$a_1 + 1.843\, a_2 = 61.035$$

$$1.3598\, a_2 = 30.384$$

$$a_2 = 22.34$$

Substituting a_2 value in equation (16),

$$\Rightarrow \quad \mathrm{a}_1 + 1.843(22.34) = 61.035$$

$$\mathrm{a}_1 + 41.173 = 61.035$$

$$\Rightarrow \quad \mathrm{a}_1 = 19.862$$

Hence the trial function is, $y = 19.862\,(x - x^3) + 22.34(x - x^5)$

Result:	a_1	a_2
(i) Point collocation:	18.53	25
(ii) Subdomain collocation:	18.50	22.23
(iii) Least squares method:	21.11	20.28
(iv) Galerkin's method:	19.862	22.34

Example 1.16

The following differential equation is available for a physical phenomenon:

$$\frac{d^2y}{dx^2} - 10\,x^2 = 5;\ 0 \le x \le 1$$

The boundary conditions are: y(0) = 0

y(1) = 0

By using Galerkin's method of weighted residuals to find an approximate solution of the above differential equation and also compare with exact solution.

Given: Differential equation,

$$\frac{d^2y}{dx^2} - 10\,x^2 = 5 \qquad ..(1)$$

boundary conditions are: y(0) = 0

$$y(1) = 0$$

To find : Approximate solution by using Galerkin's method.

Solution: We know that, a single trial function, which satisfies the stated boundary condition is,

$$y = a_1 x(x-1) = a_1(x^2 - x)$$

$$\Rightarrow \qquad \frac{dy}{dx} = a_1(2x-1)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = a_1(2-0)$$

$$\frac{d^2y}{dx^2} = 2a_1$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation (1),

$\Rightarrow$ Residual, $\qquad R = 2a_1 - 10\,x^2 - 5$

In Galerkin's method, the trial function itself is considered as the weighting function, w_i.

$$\Rightarrow \qquad \int_0^1 w_i\, R\, dx = 0 \qquad \ldots(2)$$

Here, the trial function is, $\ y = w_i = a_1 x(x-1)$

Substitute w_i *and* R Values in equation (2),

$$\Rightarrow \quad \int_{1/2}^{1} a_1 x(x-1) \times (2\,a_1 - 10\,x^2 - 5)\, dx = 0$$

$$\Rightarrow \quad \mathrm{a}_1 \int_{1/2}^{1} [2\,a_1 x^2 - 10\,x^4 - 5\,x^2 - 2\,a_1 x + 10\,x^3 + 5\,x] dx = 0$$

$$\Rightarrow \quad 2\,a_1 \left[\frac{x^3}{3}\right]_0^1 - 10\left[\frac{x^5}{5}\right]_0^1 - 5\left[\frac{x^3}{3}\right]_0^1 - 2\,a_1\left[\frac{x^2}{2}\right]_0^1 + 10\left[\frac{x^4}{4}\right]_0^1 + 5\left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\Rightarrow \ 2\, a_1[1-0] - \frac{10}{5}[1-0] - \frac{5}{3}[1-0] - \frac{2\, a_1}{2}[1-0] + \frac{10}{4}\ [1-0] + \frac{5}{2}[1-0] = 0$$

$$\Rightarrow \ \frac{2\, \mathrm{a}_1}{3} - 2 - 1.666 - a_1 + 2.5 + 2.5 = 0$$

$$\Rightarrow \qquad 0.666\, \mathrm{a}_1 - a_1 + 1.3342 = 0$$

$$\Rightarrow \qquad -0.334\, \mathrm{a}_1 = -1.334$$

$$\Rightarrow \qquad \mathrm{a}_1 = 4$$

So, the approximate solution is obtained as,

$$y = 4\, x(x - 1) \qquad \ldots(3)$$

we can compare this result with exact solution.

$$\frac{d^2y}{dx^2} = 10\, x^2 + 5\ (given)$$

$$\Rightarrow \qquad \frac{dy}{dx} = \int \frac{d^2y}{dx^2} = \frac{10\, x^3}{3} + 5\, x + c_1$$

$$\Rightarrow \qquad y = \int \frac{dy}{dx} = \frac{10\, x^4}{3\, \times 4} + \frac{5\, x^2}{2} + c_1 x + c_2$$

$$\Rightarrow \qquad y = 0.833\, x^4 + 2.5\, x^2 + c_1 x + c_2 \qquad \ldots(4)$$

Apply boundary conditions, when $x = 0, y = 0$

(4) $\Rightarrow \qquad 0 = c_2$

$\Rightarrow \qquad c_2 = 0$

When $x = 1, y = 0$

(4) $\Rightarrow \ 0 = 0.833 + 2.5 + c_1 + c_2$

$0 = 3.333 + c_1 + 0 \qquad [\because c_2 = 0]$

$\Rightarrow \qquad c_1 = -3.333$

Substitute c_1 and c_2 values in equation (4),

$$\Rightarrow \quad y = 0.833\, x^4 + 2.5\, x^2 - 3.333\, x \qquad \text{...(5)}$$

Result: Approximate solution, $y = 4\, x\, (x - 1)$

Exact solution, $y = 0.833\, x^4 + 2.5\, x^2 - 3.333\, x$

Example 1.17

The following differential equation of a physical phenomenon is given by

$$\frac{d^2y}{dx^2} - 10\, x^2 = 5$$

Obtain two term Galerkin solution by using the trial functions:

$$N_1(x) = x(x-1); N_2(x) = x^2(x-1); 0 \le x \le 1$$

The boundary conditions are: y(0) = 0

y(1) = 0

Given: Differential equation,

$$\frac{d^2y}{dx^2} - 10\, x^2 = 5 \qquad \ldots(1)$$

Trial functions, $N_1(x) = x(x-1); N_2(x) = x^2(x-1)$

$$\Rightarrow \quad y = a_1\, x\, (x-1) + a_2\, x^2(x-1) \qquad \ldots(1)$$

boundary conditions are: y(0) = 0

y(1) = 0

To find: Approximate solution by using Galerkin's method.

Solution: First we have to verify, whether the trial function satisfies the boundary conditions or not.

Trial function $y = a_1\, x\, (x-1) + a_2\, x^2(x-1)$

When $x = 0, y = 0$

$x = 1, y = 0$

Hence its satisfies the boundary conditions.

Residual, R: $y = a_1 x(x-1) + a_1 x^2(x-1)$

$$y = a_1(x^2 - x) + a_2(x^3 - x^2)$$

$$\Rightarrow \qquad \frac{dy}{dx} = a_1(2x - 1) + a_2(3\,x^2 - 2\,x)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = a_1(2 - 0) + a_2(6x - 2)$$

$$\frac{d^2y}{dx^2} = 2a_1 + 6\,a_2 x - 2\,a_2$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation (1),

$\Rightarrow$ Residual, $\qquad R = 2a_1 + 6\,a_2\,x - 2\,a_2 - 10\,x^2 - 5 \qquad \ldots(3)$

In Galerkin's method, the trial function itself is considered as the weighting function, w_i.

$$\Rightarrow \qquad \int_0^1 w_i\,R\,dx = 0$$

Or we can write,

$$\int_0^1 x(x - 1)(2\,a_1 + 6\,a_2\,x - 2\,a_2 - 10\,x^2 - 5)\,dx = 0 \qquad \ldots(4)$$

and

$$\int_0^1 x(x - 1)(2\,a_1 + 6\,a_2\,x - 2\,a_2 - 10\,x^2 - 5)\,dx = 0 \qquad \ldots(5)$$

Solving equation (4),

$$\int_0^1 (x^2 - x)(2\,a_1 + 6\,a_2\,x - 2\,a_2 - 10\,x^2 - 5)\,dx = 0$$

$$\Rightarrow \int_0^1 [2\,a_1 x^2 + 6a_2 x^3 - 2a_2 x^2 - 10x^4 - 5x^2 - 2\,a_1 x - 6a_2 x^2 + 2\,a_2 x + 10x^3$$
$$+ 5x]dx = 0$$

$$\Rightarrow 2\,a_1 \left[\frac{x^3}{3}\right]_0^1 + 6\,a_2 \left[\frac{x^4}{4}\right]_0^1 - 2\,a_2 \left[\frac{x^3}{3}\right]_0^1 - 10 \left[\frac{x^5}{5}\right]_0^1 - 5 \left[\frac{x^3}{3}\right]_0^1 - 2a_1 \left[\frac{x^2}{2}\right]_0^1$$
$$- 6\,a_2 \left[\frac{x^3}{3}\right]_0^1 + 2\,a_2 \left[\frac{x^2}{2}\right]_0^1 + 10 \left[\frac{x^4}{4}\right]_0^1 + 5 \left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\Rightarrow \frac{2\,a_1}{3} + \frac{6\,a_2}{4} - \frac{2\,a_2}{3} - \frac{10}{5} - \frac{5}{3} - \frac{2\,a_1}{2} - \frac{6\,a_1}{3} + \frac{2\,a_1}{2} + \frac{10}{4} + \frac{5}{2} = 0$$

$$\Rightarrow 0.666\,\mathrm{a}_1 + 1.5a_2 - 0.666a_2 - 2 - 1.666 - a_1 - 2\,a_2 + a_2 + 2.5 + 2.5 = 0$$

$$\Rightarrow 0.333\,\mathrm{a}_1 - 0.1666\,a_2 + 1.334 = 0$$

$$\Rightarrow -\,0.333\,\mathrm{a}_1 + 0.1666\,a_2 = 1.334$$

$$\Rightarrow \mathrm{a}_1 + 0.50\,a_2 = 4 \qquad ..(6)$$

Solving equation (5),

$$\int_0^1 (x^2 - x)(2\,a_1 + 6\,a_2\,x - 2\,a_2 - 10\,x^2 - 5)\,dx = 0$$

$$\Rightarrow \int_0^1 (x^3 - x^2)(2\,a_1 + 6\,a_2\,x - 2\,a_2 - 10\,x^2 - 5)\,dx = 0$$

$$\Rightarrow \int_0^1 2\,a_1 x^3 + 6a_2 x^4 - 2a_2 x^3 - 10x^5 - 5x^3 - 2\,a_1 x^2 - 6a_2 x^3 + 2\,a_2 x^2 + 10x^4$$
$$+ 5\,x^2 = 0$$

$$\Rightarrow\ 2\,a_1\left[\frac{x^4}{4}\right]_0^1+6\,a_2\left[\frac{x^5}{5}\right]_0^1-2\,a_2\left[\frac{x^4}{4}\right]_0^1-10\left[\frac{x^6}{6}\right]_0^1-5\left[\frac{x^4}{4}\right]_0^1-2a_1\left[\frac{x^3}{3}\right]_0^1$$

$$-\,6\,a_2\left[\frac{x^4}{4}\right]_0^1+2\,a_2\left[\frac{x^3}{3}\right]_0^1+10\left[\frac{x^5}{5}\right]_0^1+5\left[\frac{x^3}{3}\right]_0^1=0$$

$$\Rightarrow\ \frac{2\,a_1}{4}+\frac{6\,a_2}{5}-\frac{2\,a_2}{4}-\frac{10}{6}-\frac{5}{4}-\frac{2\,a_1}{3}-\frac{6\,a_1}{4}+\frac{2\,a_1}{3}+\frac{10}{5}+\frac{5}{3}=0$$

$$\Rightarrow\ 0.5\,\mathrm{a}_1+1.2a_2-0.5a_2-1.666-1.25-0.666a_1-1.5a_2+0.666a_2+2$$
$$+\,1.666=0$$

$$\Rightarrow\ -\,0.166\,\mathrm{a}_1-0.133\,a_2+0.75=0$$

$$\Rightarrow\ -\,0.166\,\mathrm{a}_1+0.133\,a_2=0.75$$

$$\Rightarrow\ \mathrm{a}_1+0.8012\,a_2=4.518 \qquad \ldots(7)$$

Solving equation (6) and (7)

$$-a_1-0.50\,a_2=-4$$
$$a_1+0.8012\,a_2=4.518$$
$$0.3012\,a_2=0.518$$
$$\Rightarrow\qquad a_2=1.719$$

Substituting a_2 value in equation (7),

$$\Rightarrow\qquad \mathrm{a}_1+0.8012(1.719)=4.518$$
$$\mathrm{a}_1=3.140$$

So, the two term approximate solution is,

$$y=3.140\,x(x-1)+1.719\,x^2(x-1)$$
$$=3.140\,x^2-3.140\,x+1.719\,x^3-1.719\,x^2$$
$$y\ =1.719\,x^3+1.421\,x^2-3.140\,x$$

Result: The two term Galerkin's approximate solution is,

$$y\ =1.719\,x^3+1.421\,x^2-3.140\,x$$

Example 1.18

The differential equation of a physical phenomenon is given by

$$\frac{d^2y}{dx^2} + y = 4x, 0 \ \leq x \leq 1$$

The boundary conditions are: y(0) = 0

y(1) = 1

Obtain two term approximate solution by using Galerkin method of weighted residuals.

Given: Differential equation,

$$\boldsymbol{\frac{d^2y}{dx^2} + y = 4x, 0 \ \leq x \leq 1} \qquad \ldots(1)$$

Boundary conditions are: y(0) = 0

y(1) = 1

To find: One term approximate solution by using Galerkin's method.

Solution: Here, the boundary conditions are not homogeneous. So, we assume a trial function as,

$$y = a_1\, x\, (x - 1) + x$$

First we have to verify, whether the trial function satisfies the boundary conditions or not.

$$y = a_1\, x\, (x - 1) + x \qquad \ldots(2)$$

When $x = 0, y = 0$

$x = 1, y = 1$

Hence its satisfies the boundary conditions.

Residual, R: $y = a_1 x(x-1) + x = a_1(x^2 - x) + x$

$$\Rightarrow \qquad \frac{dy}{dx} = a_1(2x - 1) + 1$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = a_1(2)$$

$$\frac{d^2y}{dx^2} = 2a_1$$

Substitute $\frac{d^2y}{dx^2}$ value in given differential equation.

(1)$\Rightarrow$ $$2\,a_1 + y = 4\,x$$

Substitute y value,

$\Rightarrow$ $$2\,a_1 + a_1x(x-1) + x = 4\,x$$

$\Rightarrow Residual,$ $$R = 2\,a_1 + a_1x(x-1) + x - 4\,x$$

In Galerkin's method, the trial function itself is considered as the weighting function, w_i.

$\Rightarrow$ $$\int_0^1 w_i\,R\,dx = 0 \qquad \text{... (3)}$$

Here, the trial function is, $y = w_i = x(x-1)$

Substitute w_i and R values in equation (3),

$$\Rightarrow \int_0^1 x(x-1)[2\,a_1 + a_1\,x(x-1) + x - 4x]\,dx = 0$$

$$\Rightarrow \int_0^1 x(x-1)[2\,a_1 + a_1\,x^2 - a_1x + x - 4x]\,dx = 0$$

$$\Rightarrow \int_0^1 (x^2 - x)[2\,a_1 + a_1\,x^2 - a_1x - 3x]\,dx = 0$$

$$\Rightarrow \int_0^1 [2\,a_1x^2 + a_1x^4 - a_1x^3 - 3x^3 - 2\,a_1x - a_1x^3 + a_1x^2 + 3\,x^2]dx = 0$$

$$\Rightarrow 2\,a_1\left[\frac{x^3}{3}\right]_0^1 + a_1\left[\frac{x^5}{5}\right]_0^1 - a_1\left[\frac{x^4}{4}\right]_0^1 - 3\left[\frac{x^4}{4}\right]_0^1 - 2a_1\left[\frac{x^2}{2}\right]_0^1 - a_1\left[\frac{x^4}{4}\right]_0^1$$
$$- a_1\left[\frac{x^4}{4}\right]_0^1 + a_1\left[\frac{x^3}{3}\right]_0^1 + 3\left[\frac{x^3}{3}\right]_0^1 = 0$$

$$\Rightarrow \frac{2\,a_1}{3} + \frac{a_1}{5} - \frac{a_1}{4} - \frac{3}{4} - \frac{2\,a_1}{2} - \frac{a_1}{4} + \frac{a_1}{3} + \frac{3}{3} = 0$$

$$\Rightarrow 0.666\ a_1 + 0.2a_2 - 0.25a_1 - 0.75 - a_1 - 0.25\ a_1 - 0.333\ a_1 + 1 = 0$$

$$-0.301\ a_1 = -0.25$$

$$a_1 = 0.830$$

So, the one term approximate solution is,

$$y = 0.830\ x(x-1) + x$$

$$= 0.830\ x^2 - 0.830\ x + x$$

$$y\ = 0.830\ x^2 + 0.17x$$

Result: The two term Galerkin's approximate solution is,

$$y\ = 0.830\ x^2 + 0.17\ x$$

Example 1.19

Find the deflection at the centre of a simply supported beam of span length 'l' subjected to uniformly distributed load throughout its length as shown in Fig.(i0, using (a) point collocation method, (b) Sub-domain method, (c) Least squares method, and (d) Galerkin's method.

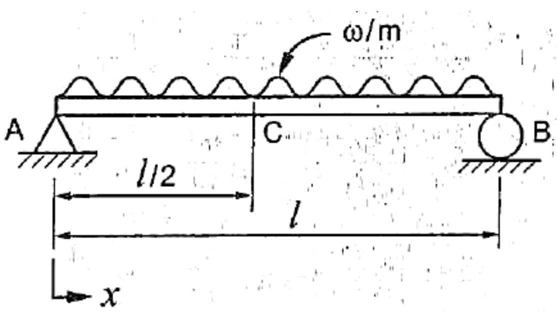
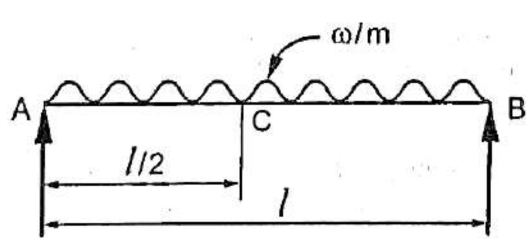


Fig. (i)

Given:



To find: Deflection at the centre "C" by using

(i) Point collocation method

(ii) Subdomain collocation method

(iii) Least squares method

(iv) Galerkin's method

Solution: The differential equation governing the deflection of beam subjected to uniformly distributed load is given by,

$$E\,I\frac{d^4y}{dx^4} - \omega = 0, 0 \le x \le 1 \qquad \text{... (1)}$$

The boundary conditions are $y = 0$ *at* $x = 0$ and $x = 1$, where y is the deflection,

$$E\,I\frac{d^4y}{dx^4} = 0 \text{ } at \text{ } x = 0 \text{ } and \text{ } x = 1$$

Where, $$E\,I\frac{d^4y}{dx^4} = M(Bending\ moment)$$

E → Young's modulus.

I → Moment of inertia of the beam.

Let us select the trial function for deflection as,

$$y = a\, sin\frac{\pi x}{l} \qquad \text{... (2)}$$

Hence it satisfies the boundary conditions,

$$\Rightarrow \qquad \frac{dy}{dx} = a\frac{\pi}{l}\,.\cos\frac{\pi\, x}{l}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\,.\sin\frac{\pi\, x}{l}$$

$$\Rightarrow \qquad \frac{d^3y}{dx^3} = -a\frac{\pi^3}{l^3}\,.\cos\frac{\pi\, x}{l}$$

$$\Rightarrow \qquad \frac{d^4y}{dx^4} = a\frac{\pi^4}{l^4}\,.\sin\frac{\pi\, x}{l} \qquad \text{... (3)}$$

Substituting the equation (3) in the governing equation (1),

$$E\ I\left(a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}\right)-\omega=0$$

Take, Residual, $$R=E\ I\ a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}-\ \omega$$

(a) Point collocation method:

In this method, the residuals are set to zero.

$$\Rightarrow \qquad R=E\ I\ a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}-\ \omega=0$$

$$E\ I\ a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}=\ \omega$$

To get maximum deflection, take $x=\frac{1}{2}$ (i.e., at the center of beam)

Then, $$E\ I\ a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}\left(\frac{l}{2}\right)=\ \omega$$

$$E\ I\ a\frac{\pi^4}{l^4}=\ \omega \qquad \left[\because\sin\frac{\pi}{2}=1\right]$$

$$a=\frac{\omega\ l^4}{\pi^4\ E\ I}$$

Substitute "a" value in the trial function equation (2),

$$y=\frac{\omega\ l^4}{\pi^4\ E\ I}\sin\frac{\pi\ x}{l}$$

$At\ x=\frac{1}{2}\Rightarrow$ $$y_{max}=\frac{\omega\ l^4}{\pi^4\ E\ I}\sin\frac{\pi}{l}\left(\frac{l}{2}\right)$$

$$y_{max}=\frac{\omega\ l^4}{\pi^4\ E\ I} \qquad \left[\because\sin\frac{\pi}{2}=1\right]$$

$$y_{max}=\frac{\omega\ l^4}{97.4\ E\ I}$$

(b) Sub-domain collocation method:

In this method, the integral of the residual over the sub-domain is set to zero.

$$\int_0^1 R\,dx = 0$$

Substitute R value,

$$\Rightarrow \qquad \int_0^1 \left(a\,E\,I\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - \omega\right)dx = 0$$

$$\Rightarrow \qquad \left[a\,E\,I\frac{\pi^4}{l^4}\left(\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right) - \omega\,x\right]_0^l = 0$$

$$\Rightarrow \qquad \left[a\,E\,I\frac{\pi^4}{l^4}\left(-\cos\frac{\pi x}{l}\right)\left(\frac{\pi}{l}\right) - \omega\,x\right]_0^l = 0$$

$$\Rightarrow \qquad -\,a\,E\,I\frac{\pi^3}{l^3}(\cos\pi - \cos 0) - \omega\,l = 0$$

$$\Rightarrow \qquad -\,a\,E\,I\frac{\pi^3}{l^3}(-1-1) = \omega\,l$$

$$[\because cos\pi = -1\,;cos\;0 = 1]$$

$$\Rightarrow \qquad a = \frac{\omega\,l^4}{2\,\pi^3 E\,I} = \frac{\omega\,l^4}{62\,E\,I}$$

Substitute “a” value in the trial function equation (2),

$$y = \frac{\omega\,l^4}{62E\,I}sin\,\frac{\pi x}{l}$$

$$At\;x = \frac{1}{2}, \qquad y_{max} = \frac{\omega\;l^4}{62\,E\,I}\sin\frac{\pi}{l}\left(\frac{l}{2}\right)$$

$$y_{max} = \frac{\omega\;l^4}{62\,E\,I}$$

(c) Least squares method:

In this method, the functional

$$I = \int_0^1 R^2 \, dx \; is \; minimum.$$

$$I = \int_0^1 \left(a\,E\,I \frac{\pi^4}{l^4} \sin\frac{\pi x}{l} - \omega \right)^2 dx$$

$$= \int_0^1 \left[a^2\,E^2\,I^2 \frac{\pi^8}{l^8} sin^2 \frac{\pi x}{l} + \omega^2 - 2\,a\,E\,I\omega\,\frac{\pi^4}{l^4} sin\,\frac{\pi x}{l} \right] dx$$

$$= \int_0^1 \left[a^2\,E^2 \frac{I^2 \pi^8}{l^8} \left(\frac{1 - \;cos\left(\frac{2\pi x}{l}\right)}{2} \right) + \omega^2 \right.$$

$$\left. - 2\,a\,E\,I\omega\,\frac{\pi^4}{l^4} sin\,\frac{\pi x}{l} \right] dx$$

$$\left[\because \sin^2 x = \frac{1 - \cos 2\,x}{2} \Rightarrow \sin^2 \frac{\pi x}{l} = \frac{1 - cos\left(\frac{2\pi x}{l}\right)}{2} \right]$$

$$= \left[a^2\,E^2 I^2 \frac{\pi^8}{l^8} \left(\frac{1}{2}\,x - \sin\frac{2\pi x}{l}\left(\frac{l}{2\pi}\right)\right) + \omega^2 x \right.$$

$$\left. - 2\,a\,E\,I\omega\,\frac{\pi^4}{l^4}\left(-\cos\frac{\pi x}{l}\left(\frac{l}{\pi}\right)\right) \right]_0^l$$

$$= a^2\,E^2 I^2 \frac{\pi^8}{l^8}\left[\frac{1}{2}\,l - \frac{1}{2\pi}(\sin 2\pi - \sin 0)\right] + \omega^2 l$$

$$- 2\,a\,E\,I\omega\,\frac{\pi^4}{l^4} . \frac{l}{\pi}(-\cos\pi - cos 0)$$

$$[\because sin\,2\pi\; = 0; \sin 0 = 0; \cos\pi = 0\,; \cos 0 = 1\,]$$

$$= a^2\,E^2 I^2 \frac{\pi^8}{l^2} . \frac{l}{2} + \omega^2 l - 2\,a\,E\,I\omega\,\frac{\pi^3}{l^3}(-1 - 1)$$

$$= \frac{a^2\, E^2 I^2 \frac{\pi^8}{l^2}}{2\, l^7} + \omega^2 l - 4\, a\, E\, I\omega\, \frac{\pi^3}{l^3}$$

$Now,$ $$\frac{\partial I}{\partial a} = 0$$

$$= \frac{2\, a\, E^2 I^2\, \pi^8}{2\, l^7} = 4\, E\, I\omega\, \frac{\pi^3}{l^3}$$

$$\frac{2\, a\, E^2 I^2\, \pi^8}{l^7} = 4\, E\, I\omega\, \frac{\pi^3}{l^3}$$

$\therefore$ $$a = \frac{4\, \omega\, l^4}{\pi^5 E\, I}$$

Hence the trial function,

$$y = \frac{4\, \omega\, l^4}{\pi^5 E\, I} \sin\frac{\pi x}{l}$$

At x $= \frac{1}{2}$, maximum defelction,

$$y_{max} = \frac{4\, \omega\, l^4}{\pi^5 E\, I} \sin\frac{\pi x}{l}\left(\frac{l}{2}\right) \qquad \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$y_{max} = \frac{\omega\, l^4}{76.5\, E\, I}$$

(d) Galerkin's method:

In this method,

$$\int_0^1 y\, R\, dx = 0.$$

$$\Rightarrow \int_0^1 \left[\left(a\, sin\frac{\pi x}{l}\right)\left(a\, E\, I\, \frac{\pi^4}{l^4}\, sin\frac{\pi x}{l} - \omega\right)\right] dx = 0$$

$$\Rightarrow \int_0^1 \left[a^2 E\, I\, \frac{\pi^4}{l^4} \sin^2\frac{\pi x}{l} - a\, \omega \sin\frac{\pi x}{l}\right] dx = 0$$

$$= \int_0^1 \left[a^2\, E\, I \frac{\pi^4}{l^4} \left[\frac{1}{2} \left(1 - \cos \frac{2\pi x}{l} \right) \right] - a\, \omega\, sin\, \frac{\pi x}{l} \right] \;\; dx = 0$$

$$\Rightarrow \left[a^2\, E\, I \frac{\pi^4}{l^4} \left[\frac{1}{2} \left\{ x - \left(\frac{1}{2\,\pi} \right) \sin \frac{2\pi x}{l} \right\} \right] + a\, \omega \frac{l}{\pi} \left\{ \cos \frac{\pi x}{l} \right\} \right]_0^l = 0$$

$$a^2\, E\, I \frac{\pi^4}{l^4} \left(\frac{1}{2} \right) - 2\, a\, \omega \left(\frac{l}{\pi} \right) = 0$$

$$\therefore \quad a = \frac{2\,\omega\, l}{\pi} . \frac{2\;\, l^3}{E\, I\, \pi^4}$$

$$a = \frac{4\,\omega\, l^4}{\pi^5 E\, I}$$

Hence, the trial function,

$$y = \frac{4\,\omega\, l^4}{\pi^5 E\, I} \,.\, sin \frac{\pi\, x}{l}$$

At $x = \frac{1}{2}$, maximum deflection,

$$y_{max} = \frac{4\,\omega\, l^4}{\pi^5 E\, I}\; sin \frac{\pi\, x}{l} \left(\frac{l}{2} \right) = \frac{4\,\omega\, l^4}{\pi^5 E\, I}$$

$$y_{max} = \frac{\omega\, l^4}{76.5\; E\, I}$$

Verification: we know that, simply supported beam is subjected to uniformly distributed load, maximum deflection is,

$$y_{max} = \frac{5}{384}\;\; \frac{\omega\, l^4}{E\, I} = 0.01\;\; \frac{\omega\, l^4}{E\, I}$$

Result: Maximum deflection at $x = \frac{1}{2}$

(a) Point collocation method:

$$y_{max} = \frac{\omega\;\, l^4}{97.4\; E\, I} = 0.01 \frac{\omega\;\, l^4}{E\, I}$$

(b) Sub-domain collocation method:

$$y_{max} = \frac{\omega\ l^4}{62\ E\ I} = 0.01\frac{\omega\ l^4}{E\ I}$$

(c) Least squares method:

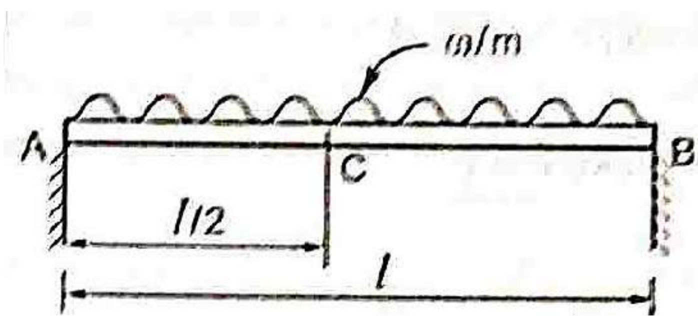
$$y_{max} = \frac{\omega\ l^4}{62\ E\ I} = 0.01\frac{\omega\ l^4}{E\ I}$$

(d) Galerkin's method:

$$y_{max} = \frac{\omega\ l^4}{76.5\ E\ I} = 0.01\frac{\omega\ l^4}{E\ I}$$

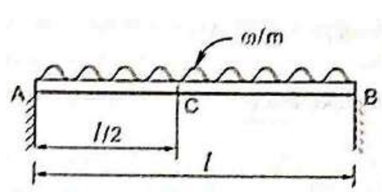
Example 1.20

Find the deflection at the centre of a clamped beam subject to uniformly distributed load throughout its length as shown in Fig. (i). use point collection method. Take trial function as $y = a(x^5 - 2lx^4 + l^2x^3)$.



Fig, (i)

Given:



Fig, (ii)

Solution: The differential equation governing the deflection of the beam subjected to udl is given by

$$EI\frac{d^4y}{dx^4} - \omega = 0, 0 \leq x \leq l \qquad \text{....}(1)$$

Since this is the clamped beam (i.e., fixed end beam), the boundary conditions are,

Deflection, y = 0 at x = 0 and x =1

Slope, $\theta = EI\dfrac{dy}{dx} = 0\ at\ x = 0\ and\ x = 1$

The trial function should satisfy the above boundary conditions

We know that,

Trial function, $y = a(x^5 - 2lx^4 + l^2x^3)$ …(2)

$$\Rightarrow \frac{dy}{dx} = a(5x^4 - 8lx^3 + 3l^2x^2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = a(20x^4 - 24lx^2 + 6l^2x)$$

$$\Rightarrow \frac{d^3y}{dx^3} = a(60x^4 - 48l + 6l^2)$$

$$\Rightarrow \frac{d^4y}{dx^4} = a(120x - 48l)$$

Substituting $\dfrac{d^4y}{dx^4}$ value in equation (1), we get

$$a\ EI\ (120x - 48l) - \omega = 0$$

Take Residual, $R = a\ EI\ (120x - 48l) - \omega$

In point collocation method, the residual is set to zero

i.e., $a\ EI\ (120x - 48l) - \omega = 0$

To get maximum deflection, take x = $\dfrac{l}{2}$

$$\Rightarrow a\ EI\left(120 \times \frac{l}{2} - 48l\right) = \omega$$

$$120a\ EIl = \omega$$

$$a\ = \frac{\omega}{120\ EIl}$$

Hence, the trial function $y = \frac{\omega}{120\ EIl}(x^5 - 2lx^4 + l^2x^3)$

At $x = \frac{l}{2}$, Maximum deflection,

$$y_{max} = \frac{\omega}{120\ EIl}\left[\left(\frac{l}{2}\right)^5 - 2l\left(\frac{l}{2}\right)^4 + l^2\left(\frac{l}{2}\right)^3\right]$$

$$y_{max} = \frac{\omega l^4}{384EI}$$

Result: Maximum deflection at $x = \frac{l}{2}$

$$y_{max} = \frac{\omega l^4}{384EI}$$

1.13. THE GENERAL WEIGHTED RESIDUAL STATEMENT

After understanding the basic technique and successfully solved a few problems, the general weighted residual statement can be written as

$$\int w_i\, R\, dx = 0\ for\ i = 1, 2, \ldots\ldots, n \qquad \ldots (1.13)$$

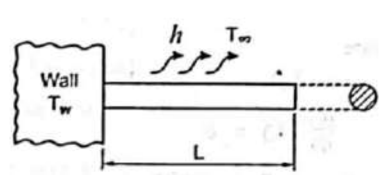
$$\text{Where } w_i = N_i \qquad \ldots (1.14)$$

The better result will be obtained by considering more terms in polynomial and trigonometric series.

SOLVED PROBLEMS- GENERAL WEIGHTED RESIDUAL METHOD

Example 1.21

Consider a 1 mm diameter, 50 mm long aluminium pin-fin as shown in Fig. (i) used to enhance the heat transfer from a surface wall maintained at 300ºC. the governing differential equation and the boundary conditions are given by.



Fig, (i)

$$K\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)$$

$$T(0) = T_w = 300^oC$$

$$\frac{\mathrm{dT}}{\mathrm{dx}}(\mathrm{L}) = 0(\text{insulated tip})$$

Where, K = Thermal conductivity

P = Perimeter

A = Cross – sectional area

h = Convective heat transfer coefficient

T_w = Wall temperature

T_∞ = Ambient temperature

Let, K = 200 W/m°C for aluminium, h = 20 W/m^2 °C, $T_\infty = 30^oC$. Estimate the temperature distribution in the fin using the Galerkin weighted residual method.

Given: Diameter, d = 1 mm = 1×10^{-3} m

Length, L = 50 = 50×10^{-3} m

Wall temperature, Tw = 300°C

Governing differential equation,

$$K\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)$$

$$T(0) = T_w = 300^oC$$

$$\frac{\mathrm{dT}}{\mathrm{dx}}(\mathrm{L}) = 0(\text{insulated tip})$$

Thermal conductivity, K = 200 W/m °C

Heat transfer coefficient, h = 20 W/m^2 °C

Ambient temperature, T_∞ = 30°C

To find: Temperature distribution using Galerkin method.

Solution: Assume a trial solution. Let,

$$T(x) = a_0 + a_1x + a_2x^2 \quad \ldots(1)$$

The boundary conditions are

$$T(0) = T_w = 300^o C \qquad \ldots(a)$$

$$\frac{\mathrm{dT}}{\mathrm{dx}}(\mathrm{L}) = 0 \qquad \ldots(b)$$

From equation (a) $\Rightarrow x = 0$, T = 300°C

Applying these values in equation (1),

$$\Rightarrow \; a_0 = 300$$

From equation (b) $\Rightarrow$

$$x = L, \frac{\mathrm{dT}}{\mathrm{dx}} = 0$$

Differentiate equation (1),

$$\frac{\mathrm{dT}}{\mathrm{dx}} = a_1 + a_2 2x \qquad \ldots(2)$$

$$0 = a_1 + a_2(2L)$$

$$a_1 = -2La_2$$

Substitute a_0 and a_1 value in equation (1),

$$T(x) = 300 + (-2La_2)x + a_2x^2$$

$$T(x) = 300 + a_2(x^2 - 2Lx) \qquad \ldots(3)$$

We know that

$$K\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)$$

$$200\frac{d^2T}{dx^2} = \frac{\pi(1 \times 10^{-3}) \times 20}{\frac{\pi}{4}(1 \times 10^{-3})^2}(T - 30) \qquad [\because P = \pi D]$$

$$\frac{d^2T}{dx^2} = 400(T - 30) \qquad \ldots(4)$$

Substitute 'T' value from equation (3)

$$\frac{d^2T}{dx^2} = 400[(300 + a_2(x^2 - 2Lx) - 30)]$$

$$= 400[270 + a_2(x^2 - 2Lx)] \quad \text{... (5)}$$

From equation (2),

$$\frac{\mathrm{dT}}{\mathrm{dx}} = a_1 + a_2 2x$$

$$\Rightarrow \frac{\mathrm{dT}}{\mathrm{dx}} = 2a_2 \quad \text{... (6)}$$

Substitute the equation (6) in equation (5),

$$2a_2 = 400[270 + a_2(x^2 - 2Lx)]$$

$$2a_2 - 400[270 + a_2(x^2 - 2Lx)] = 0$$

$$\text{Take, Residual,} \quad R = 2a_2 - 400[270 + a_2(x^2 - 2Lx)] \quad \text{... (8)}$$

To minimize the residual, take weight function as

$$\mathrm{W(x)} = \mathrm{x}^2 - 2\ \mathrm{Lx}$$

$$\Rightarrow \int_0^l (x^2 - 2Lx)R\ dx = 0 \quad \text{... (9)}$$

Substitute the equation (8) in equation (9),

$$\int_0^L (x^2 - 2Lx)[2\ a^2 - 400(270 + a_2(x^2 - 2\ L\ x)]dx = 0$$

$$\int_0^L (x^2 - 2Lx)[(2\ a^2 - 400a_2x^2 + 800a_2Lx]dx = 0$$

$$\Rightarrow \int_0^L [(2\ a^2x^2 - 108000 - 400a_2x^4 + 800a_2Lx^3 - 4a_2lx + 21600\ lx$$

$$+ 800a_2lx^3 - 1600a_2l^2x^2)]dx = 0$$

$$\Rightarrow \left(2\,a^2\frac{x^2}{3} - 108000\frac{x^3}{3} - 400a_2\frac{x^5}{5} + 800a_2 l\frac{x^4}{4} - 4a_2\frac{lx^2}{2} + 21600\,\frac{lx^2}{2}\right.$$
$$\left. + 800a_2\frac{l\,x^4}{4} - 1600a_2\frac{l^2x^3}{3}\right)_0^l = 0$$

$$\Rightarrow \left(2\,a^2\frac{l^3}{3} - 108000\frac{l^3}{3} - 400a_2\frac{1^5}{5} + 800a_2\frac{l^4}{4} - 4a_2\frac{l^2x^3}{3} + 21600\,\frac{l^3}{2}\right.$$
$$\left. + 800a_2\frac{l^5}{4} - 1600a_2\frac{l^5}{3}\right) = 0$$

$$\Rightarrow \frac{2a^2}{3} - \frac{10800}{3} - 400\frac{a^2l^2}{5} + 800\frac{a^2l^2}{4} - \frac{4a^2}{2} + \frac{21600}{2} + 800a_2\frac{l^2}{4}$$
$$- 1600a_2\frac{l^2}{3} = 0$$

$$\Rightarrow a_2[0.6667 - 80\,l^2 + 200l^2 - 2 + 200\,l^2 - \frac{1600l^2}{3} = -72000$$

Substitute L = 50 × 10^{-3} m

$$\Rightarrow a_2[0.6667 - 0.2 + 0.5 - 2 + 0.5 - 1.3333 = -72000$$
$$\Rightarrow a_2[-1.8666] = -72000$$
$$a_2 = 38572.80$$

Galerkin solution, T(x) = 300 + 38572.80 (x^2 – 2 L x)

Result: Galerkin solution, T(x) = 300 + 38572.80 (x^2 – 2 L x)

1.14 VARIATIONAL (WEAK) FORM OF THE WEIGHTED RESUDUAL STATEMENT

We know that the general weighted residual statement is,

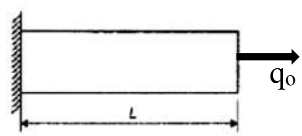
$$\int w_i\,R\,dx = 0 \qquad \text{... (1.15)}$$

In this variational method, integration is carried out by parts. It reduces the continuity requirement on the trial function assumed in the solution. So, it is referred to as the week form. In this method, it is possible to have a wider choice of trial functions.

1.15 COMPARISON OF DIFFERENTIAL EQUATION, WEIGHTED RESIDUAL STATEMENT AND WEAK FORMULATION OF WEIGHTED RESUDUAL STATEMENT

1.15.1 Differential Equation

Consider a uniform rod subjected to uniform axial load q_0 as shown in Fig. 1.26



Fig, 1.26 Uniform rod

The deformation of the bar is governed by the differential equation,

$$A\,E\,\frac{d^2u}{dx^2} + q_0 = 0 \qquad \text{... (1.16)}$$

With the boundary conditions,

$$u\,(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=1} = P_1 \qquad \text{... (1.17)}$$

1.15.2 Weighted Residual Statement

In order to find the solution for the aforementioned problem, the weighted residual statement can be developed as follows:

$$\int_0^l w(x)\left[A\,E\,\frac{d^2u}{dx^2} + q_0\right]dx = 0 \qquad \text{... (1.18)}$$

With the boundary conditions,

$$u\,(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l \qquad \text{... (1.19)}$$

1.15.3 Observations on the Weighted Residual Statement

- ✓ Weighted residual statement can be developed for any form of differential equations like linear, non-linear, ordinary, partial, etc.
- ✓ The weighted residual statement is developed only for differential equation and it is not suitable for boundary conditions.
- ✓ The trail solution should satisfy all the boundary conditions and it should be differentiable as many times as needed in the original differential equation.

1.15.4 Weak form of Weighted Residual Statement

By performing integration by parts, the weak form of weighted residual statement of the aforementioned problem is obtained as follows:

$$\left[w(x) A\, E \frac{du}{dx}\right]_0^l - \int_0^l A\, E \frac{du}{dx} . \frac{dw}{dx} . dx + \int_0^l w(x)\, q dx = 0 \qquad \text{... (1.20)}$$

With the boundary conditions,

$$u\,(0) = 0$$

$$A\, E \left.\frac{du}{dx}\right|_{x=l} = P_l$$

When we apply the boundary conditions, the equation (1.20) reduces to

$$\int_0^l A\, E \frac{du}{dx} . \frac{dw}{dx} . dx = \int_0^l w(x)\, q dx + w(l) P_l \qquad \text{... (1.21)}$$

1.15.5 Observations on the Weak Form

- ✓ A much wider choice of trial functions can be used.
- ✓ The weak form can be developed for any higher order differential equation.
- ✓ Natural boundary conditions are directly applied in the differential equation.
- ✓ The trial solution should satisfy the essential boundary conditions.

1.16 PRINCIPLE OF STATIONARY TOTAL POTENTIAL (PSTP)

1.16.1. Potential Energy in Elastic Bodies

Potential energy is the capacity to do the work by the forces acting on deformable bodies. The forces acting on a body may be classified as external forces and internal forces. External forces are the applied loads while internal forces are the stresses developed in the body. Hence the total potential energy is the sum of internal and external potential energies.

Consider a spring-mass system as shown in Fig. 1.27. Let its stiffness (load per unit deflection) be k and length L. Due to a force P let it extend by u.

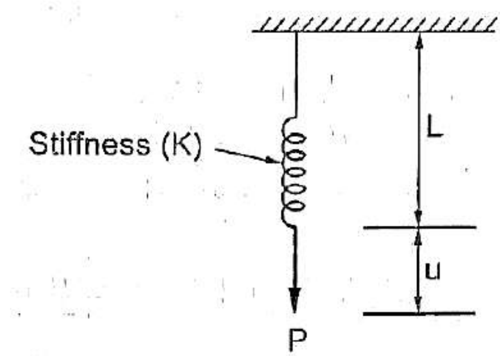


Fig. 1.27

The load P moves down by distance u. Hence it loses its capacity to do work by P u. the external potential energy in this case is given by,

$$H = - P\, u \qquad \ldots(1.22)$$

When the load has reached equilibrium position after extension of spring by u, the force in spring is K u. But when extension was zero the resisting force was also zero.

Hence, the average force $= \frac{Ku}{2}$

The energy stored in the spring due to strain

$$= \text{Average force} \times \text{Deflection}$$

$$= \frac{Ku}{2} \times u$$

$$= \frac{1}{2} \mathrm{K}\, u^2$$

∴ Total potential energy in the spring,

$$\pi = \frac{1}{2} \mathrm{K}\, u^2 - P\, u \qquad \text{... (1.23)}$$

1.16.2 Principle of Minimum Potential Energy

From the expression for total potential energy,

$$\pi = U + H \qquad \text{... (1.24)}$$

We know that, Potential energy of external force H is equal but opposite to total virtual work done by external force.

Thus, $H = -H_e$

$$\therefore \delta H = -\delta H_e$$

$$\Rightarrow \delta\pi = \delta U - \delta H \qquad \text{.... (1.25)}$$

In, principle of virtual work, $\delta U = \delta H$

$$\therefore \delta\pi = 0$$

Hence, we can conclude that a deformable body is in equilibrium when the potential energy is having stationary value.

Hence, the principal of minimum potential energy states "Among all the displacement equations that internal compatibility and the boundary condition those that also satisfy the equations of equilibrium make the potential energy a minimum is a stable system."

1.17 SOLVED PROBLESM – POTENTIAL ENERGY APPROCH

Example 1.22

The spring assembly is shown in Fig (i). assemble the finite element equation by using direct approach and potential energy approach.

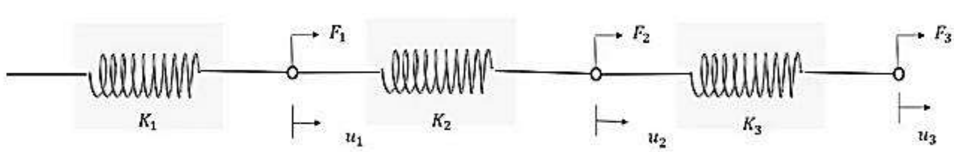


Fig. (i)

Given:

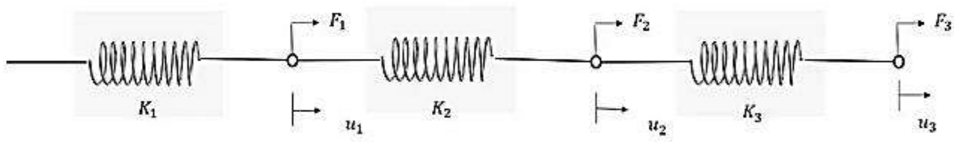


Fig. (ii)

To find: Global stiffness matrix for the spring system.

Solution: Consider the free body diagram of nodes 1, 2 and 3 as shown in Fig. (iii). Let the displacement of nodes by u_1, u_2 and u_3. The extension of spring 1, 2 and 3 are,

We know that, Displacement,

$$\delta_1 = u_1, \delta_2 = u_2 - u_1 , \delta_3 = u_3 - -u_2 \qquad \text{... (1)}$$

The equilibrium equations are (from Fig. (iii))

$$-k_1\delta_1 + k_2\delta_2 + F_1 = 0 \qquad \text{... (2)}$$

$$-k_2\delta_2 + k_3\delta_3 + F_2 = 0 \qquad \text{... (3)}$$

$$-k_3\delta_3 + F_3 = 0 \qquad \text{... (4)}$$

Substitute δ_1, δ_2 and δ_3 values in equations (2), (3) and (4).

Equation (2) $\Rightarrow$ $$-k_1u_1 + k_2(u_2 - u_1) = -F_1$$

$$k_1u_1 - (u_2 - u_1) = -F_1$$

$$(k_1 + k_2)u_1 - k_2u_2 = -F_1 \qquad \text{...(5)}$$

Equation (3) $\Rightarrow$ $-k_2(u_2 - u_1) + k_3(u_3 - u_2) = -F_2$

$$k_2(u_2 - u_1) - k_3(u_3 - u_2) = F_2$$

$$-k_2u_1 + (k_2 + k_3)u_2 - k_3u_3 = -F_2 \qquad \text{... (6)}$$

Equation (4) $\Rightarrow$ $$-k_3(u_3 - u_2) = -F_3$$

$$k_3(u_3 - u_2) = F_3$$

$$-k_3u_2 + k_3u_3 = F_3 \quad \text{.... (7)}$$

Arranging equations (5), (6) and (7) in matrix form

$$\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \text{... (8)}$$

Now, let us see the potential energy approach. Total potential energy in the system is,

$$\pi = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 + \frac{1}{2}k_3\delta_3^2 - F_1u_1 - F_2u_2 - F_3u_3$$

$$= \frac{1}{2}k_1u_1^2 + \frac{1}{2}k_2(u_2-u_1)^2 + \frac{1}{2}k_3(u_3-u_2)^2 - F_1u_1 - F_2u_2 - F_3u_3$$

Apply, $\dfrac{\partial \pi}{\partial u_1} = 0$

$$\Rightarrow \quad -k_1u_1 + k_2(u_2-u_1)(-1) - F_1 = 0$$

$$-k_1u_1 - k_2(u_2-u_1) - F_1 = 0$$

$$(k_1+k_2)u_1 - k_2u_2 = F_1 \quad \text{...(9)}$$

Apply, $\dfrac{\partial \pi}{\partial u_2} = 0$

$$\Rightarrow \quad k_2(u_2-u_1) + k_3(u_3-u_2)(-1) = F_2$$

$$-k_2u_1 - (k_2+k_3)u_2 - k_3u_3 = F_2 \quad \text{... (10)}$$

Apply, $\dfrac{\partial \pi}{\partial u_3} = 0$

$$\Rightarrow \quad k_3(u_3-u_2) - F_3 = 0$$

$$-k_3u_2 + k_3u_3 = F_3 \quad \text{... (11)}$$

Equation (9), (10) and (11) in matrix form,

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \ldots (12)$$

Result: Finite element equation

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Example 1.23

Determine the displacements of nodes 1 and 2 in the spring system shown in Fig. (i). use minimum of potential energy principle to assemble equations of equilibrium.

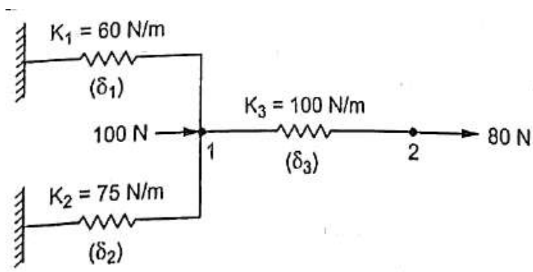


Fig. (i)

Given:

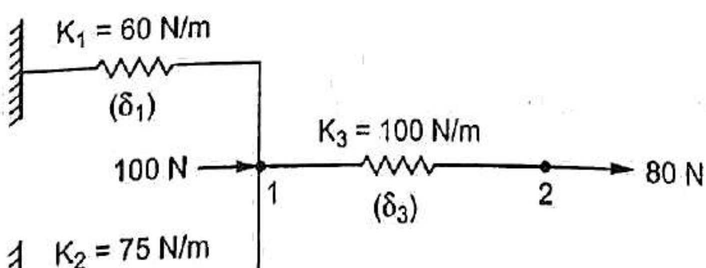


Fig. (ii)

$k_1 = 60$ N/m, $F_1 = 100$ N

$k_2 = 75$ N/m, $F_2 = 80$ N

$k_3 = 100$ N/m

To find: Displacements of nodes 1 and 2.

Solution: Let u_1 and u_2 be the displacements of nodes 1 and 2. Then the extensions of springs are,

$$\delta_1 = u_1, \delta_2 = u_2 - u_1 \, , \delta_3 = u_3 - u_2$$

We know that,

Minimum of potential energy principle,

$$\pi = \text{Strain energy} - \text{Work done}$$

$$\pi = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 + \frac{1}{2}k_3\delta_3^2 - 100u_1 - 80u_2$$

$$= \frac{1}{2}k_1u_1^2 + \frac{1}{2}k_2u_1^2 + \frac{1}{2}k_3(u_2 - u_1)^2 - 100u_1 - 80 \dots (1)$$

$\therefore$ Now, $$\frac{\partial \pi}{\partial u_1} = 0$$

$$\Rightarrow \qquad k_1u_1 + k_2u_1 + k_3(u_2 - u_1)(-1) - 100 = 0$$

$$\Rightarrow \qquad (k_1 + k_2 + k_3)u_1 - k_3u_2 = 0 \qquad \dots(2)$$

Similarly, $$\frac{\partial \pi}{\partial u_2} = 0 \Rightarrow \qquad k_3(u_2 - u_1) - 80 = 0$$

$$-k_3u_1 + k_3u_2 = 80 \qquad \dots (3)$$

Arranging equation (2) and (3) in matrix form,

$$\begin{bmatrix} k_1 + k_2 + k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

Substituting the values of k_1, $k_2$and k_3, we get,

$$\begin{bmatrix} 60 + 75 + 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

$$\begin{bmatrix} 235 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

$$235u_1 - 100u_2 = 100$$

$$-100u_1 + 100u_2 = 80$$

$$135u_1 = 180$$

$$u_1 = \frac{180}{135}$$

$$u_1 = 1.333$$

Substitute the u_1 – value in equation (5)

$$-100(1.333) + 100u_2 = 80$$

$$u_2 = 2.133$$

Result: Displacements of nodes,

$$u_1 = 1.333m$$

$$u_2 = 2.133m$$

1.18 RAYLEIGH – RITZ METHOD (VARIATIONAL APROACH)

1.18.1. Introduction

- ✓ Rayleigh – Ritz method is an integral approach method which is useful for solving complex structural problems, encountered in finite element analysis. This method is possible only if a suitable functional is available, otherwise Galerkin's method of weighted residual is used. By using this method stiffness matrices and consistent load vector can be assembled easily. This method is mostly used for solving solid mechanics problems.
- ✓ The phrase "Variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum potential energy in solid and structural mechanics, to determine the approximate solutions of the problems.
- ✓ In Rayleigh – Ritz method for continuous system we deal with the following functional.

$$\text{Potential energy}, \pi = \int_{x_1}^{x_2} f(y, y', y'')dx \qquad \text{... (1.26)}$$

- ✓ In out terminology, a functional is an integral expression that implicitly contains the governing differential equations for a particular problem/
- ✓ Total potential energy of the structure is given by,

$$\pi = \begin{Bmatrix} \text{Internal} \\ \text{Potential} \\ \text{energy} \end{Bmatrix} - \begin{Bmatrix} \text{External} \\ \text{Potential} \\ \text{energy} \end{Bmatrix}$$

= Strain Energy – Work done by External Forces

$$\pi = U - H$$

- ✓ In this method, the approximating functions must satisfy the boundary conditions and should be easy to use. Polynomials are generally used and sometimes sine and cosine terms are also used as approximating function.
- ✓ In general, any exact function can be represented as a polynomial or trigonometric series with undetermined constants as shown below.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

or

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + \cdots$$

The constants a_0, a_1, a_2 are unknowns known as Ritz parameters of the curve. When the parameters are infinite, the particular polynomial tends to match the exact value. So, the accuracy depends upon the number of parameters chosen.

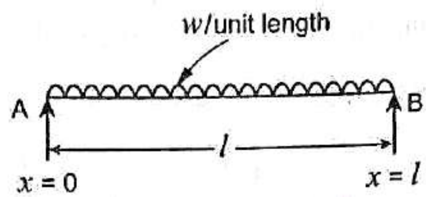
- ✓ The following two conditions must be fulfilled by the approximating function.
  1. It should satisfy the geometric boundary conditions.
  2. The function must have atleast one Ritz parameter.
- ✓ In general, a Rayleigh-Ritz solution is rarely exact except in some special simple cases, but it becomes more accurate with the use of more parameters.
- ✓ This method can be understood clearly by solving the following examples.

1.18.2 Solved Problems (on Rayleigh – Ritz Method)

Example 1.25

A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at midspan by using Rayleigh-Ritz method and compare with exact solutions.

Given:



To find: 1. Deflection and Bending moment at midspan.

2. Compare with exact solutions.

Solution: We know that, for simply supported beam, the Fourier series,

$$y = \sum_{n=1,3}^{\infty} a \sin \frac{n\pi x}{l} \text{ is the approximating function.}$$

To make this series more simple let us consider only two terms.

$$\text{Deflection, } y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l} \quad \text{... (1)}$$

Where, a_1, a_2 are Ritz Parameters.

We know that,

Total potential energy of the beam, π = U – H …(2)

Where, U → Strain energy.

H → Work done by external force.

The strain energy, U, of the beam due to bending is given by,

$$U = \frac{EI}{2} \int_0^1 \left(\frac{d^2y}{dx^2} \right)^2 dx \quad \text{... (3)}$$

$$\frac{dy}{dx} = a_1 \cos\frac{\pi x}{l} \times \left(\frac{\pi}{l}\right) + a_2 \cos\frac{3\pi x}{l} \left(\frac{3\pi}{l}\right)$$

$$\frac{dy}{dx} = \frac{a_1\pi}{l} cos\left(\frac{\pi x}{l}\right) + \frac{a_2 3\pi}{l} \cos\frac{3\pi x}{l}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a_1\pi}{l} sin\left(\frac{\pi x}{l}\right) \times \left(\frac{\pi}{l}\right) - a_2\frac{3\pi}{l} \sin\frac{3\pi x}{l} \times \left(\frac{3\pi}{l}\right)$$

$$= \frac{-\pi^2 a_1}{l} sin\left(\frac{\pi x}{l}\right) - a_2\frac{9\pi^2}{l} \sin\frac{3\pi x}{l}$$

$$\frac{d^2y}{dx^2} = \left[\frac{-a_1\pi^2}{l^2} sin\left(\frac{\pi x}{l}\right) - \frac{a_2 9\pi}{l} \sin\frac{3\pi x}{l}\right] \qquad \ldots (4)$$

Substituting $\frac{d^2y}{dx^2}$ value in equation (3),

$$\Rightarrow U = \frac{EI}{2}\int_0^l \left[\frac{-a_1\pi^2}{l^2} sin\left(\frac{\pi x}{l}\right) - \frac{a_2 9\pi^2}{l} \sin\frac{3\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2}\int_0^l \left[\frac{a_1\pi^2}{l^2} sin\left(\frac{\pi x}{l}\right) + \frac{a_2 9\pi^2}{l} \sin\frac{3\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2} \times \frac{\pi^4}{l^4}\int_0^l \left[a_1 sin\left(\frac{\pi x}{l}\right) + 9a_2 \sin\frac{3\pi x}{l}\right]^2 dx$$

$$U = \frac{EI}{2} \times \frac{\pi^4}{l^4}\int_0^l \left[a_1^2 sin^2\left(\frac{\pi x}{l}\right) + 81a_2^2 \sin^2\frac{3\pi x}{l}\right.$$

$$\left. + 2a_1 sin\left(\frac{\pi x}{l}\right) 9a_2 \sin\frac{3\pi x}{l}\right] dx$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$U = \frac{EI}{2} \times \frac{\pi^4}{l^4} \int_0^l \left[a_1^2 sin^2 \left(\frac{\pi x}{l}\right) + 81 a_2^2 \sin^2 \frac{3\pi x}{l} \right.$$

$$\left. + 18 a_1 a_2 sin \left(\frac{\pi x}{l}\right) \sin \frac{3\pi x}{l} \right] \quad dx \quad \ldots (5)$$

Since, $\int_0^l a_1^2 sin^2 \left(\frac{\pi x}{l}\right) dx = a_1^2 \int_0^l \frac{1}{2}\left(1 - \frac{\cos 2\pi x}{l}\right) dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$$= \frac{a_1^2}{2} \int_0^l \left(1 - \frac{\cos 2\pi x}{l}\right) dx$$

$$= \frac{a_1^2}{2} \left[\int_0^l dx - \int_0^l cos \frac{2\pi x}{l} \; dx \right]$$

$$= \frac{a_1^2}{2} \left[(x)_0^l - \left(\frac{sin \frac{2\pi x}{l}}{\frac{2\pi}{l}} \right)_0^l \right]$$

$$= \frac{a_1^2}{2} \left[1 - 0 - \frac{l}{2\pi} \left(sin \frac{2\pi x}{l} - \sin 0 \right) \right]$$

$$= \frac{a_1^2}{2} \left[1 - \frac{l}{2\pi} (0 - 0) \right] = \frac{a_1^2 l}{2} \qquad [\because \sin 2\pi = 0; \sin 0 = 0]$$

$$\int_0^l a_1^2 sin^2 \left(\frac{\pi x}{l}\right) dx = \frac{a_1^2 l}{2} \qquad \ldots (6)$$

Similarly,

$$\int_0^l 81 \; a_1^2 sin^2 \left(\frac{3\pi x}{l}\right) = 81 a_2^2 \int_0^l \frac{1}{2} \left(1 - \cos \frac{6\pi x}{l}\right) dx \qquad \left[\because \sin^2 x = \frac{1 - \cos 2x}{2}\right]$$

$$= \frac{81 a_2^2}{2} \left[\int_0^l dx - \int_0^l cos \frac{6\pi x}{l} \; dx \right]$$

$$= \frac{81a_2^2}{2}\left[(x)_0^l - \left(\frac{sin\frac{6\pi x}{l}}{\frac{6\pi}{l}}\right)_0^l\right]$$

$$= \frac{81a_2^2}{2}\left[1 - 0 - \frac{l}{6\pi}\left(sin\frac{6\pi x}{l} - \sin 0\right)\right]$$

$$= \frac{a_1^2}{2}\left[1 - \frac{l}{6\pi}(\sin 6\pi - \sin 0)\right] \quad [\because \sin 6\pi = 0; \sin 0 = 0]$$

$$\Rightarrow \int_0^l 81\ a_1^2 sin^2\left(\frac{3\pi x}{l}\right) = \frac{81a_2^2 l}{2} \qquad \dots(7)$$

$$\int_0^l 18\ a_1 a_2 sin\frac{\pi x}{l} sin\left(\frac{3\pi x}{l}\right) = 18\ a_1 a_2 \int_0^l sin\frac{\pi x}{l} sin\left(\frac{3\pi x}{l}\right)$$

$$= 18\ a_1 a_2 \int_0^l sin\frac{3\pi x}{l} sin\frac{\pi x}{l}$$

$$= 18\ a_1 a_2 \int_0^l \frac{1}{2}\left(cos\frac{2\pi x}{l} - cos\frac{4\pi x}{l}\right) dx$$

$$\left[\because \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}\right]$$

$$= \frac{18\ a_1 a_2}{2}\left[\int_0^l cos\frac{2\pi x}{l} dx - \int_0^l cos\frac{4\pi x}{l} dx\right]$$

$$= \frac{18\ a_1 a_2}{2}\left[\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l - \left(\frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}}\right)_0^l\right]$$

$$= 9a_1 a_2[0 - 0] = 0$$

$$[\because \sin 2\pi = 0; \sin 4\pi = 0; \sin 0 = 0]$$

$$\Rightarrow \int_0^l 18\, a_1 a_2 sin\frac{\pi x}{l} sin\left(\frac{3\pi x}{l}\right) = 0 \qquad \text{... (8)}$$

Substitute (6), (7) and (8) in equation (5),

$$(5) \Rightarrow \qquad U = \frac{EI}{2}\frac{\pi^4}{l^4}\left[\frac{a_1^2 l}{2} + \frac{81 a_2^2 l}{2} + 0\right]$$

$$U = \frac{EI}{2}\frac{\pi^4 l}{l^4}[a_1^2 + 81 a_2^2]$$

$$\text{Strain energy,} \quad \mathrm{U} = \frac{\mathrm{EI}}{2}\frac{\pi^4}{l^3}[a_1^2 + 81 a_2^2] \qquad \text{... (9)}$$

We know that,

Work done by external force,

$$H = \int_0^l \omega\, y\, dx = \int_0^l \omega\, \left(a_1 sin\frac{\pi x}{l} + a_2 sin\frac{3\pi x}{l}\right) dx$$

$$= \omega \int_0^l \left(a_1 sin\frac{\pi x}{l} + a_2 sin\frac{3\pi x}{l}\right) dx$$

$$= \omega \left[a_1 \int_0^l sin\frac{\pi x}{l} dx + a_2 \int_0^l sin\frac{3\pi x}{l} dx\right]$$

$$= \omega \left[a_1 \left(\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right)_0^l - a_2 \left(\frac{-\cos\frac{3\pi x}{l}}{\frac{3\pi}{l}}\right)_0^l\right]$$

$$= \omega \left[\frac{a_1 l}{\pi}\left(\cos\frac{\pi x}{l}\right)_0^l - \frac{a_2 l}{3\pi}\left(\cos\frac{3\pi x}{l}\right)_0^l\right]$$

$$= \omega \left[\frac{a_1 l}{\pi}[(-1) - 1] - \frac{a_2 l}{3\pi}[(-1) - 1]\right]$$

$$= \omega\left[\frac{2a_1 l}{\pi} + \frac{2a_2 l}{3\pi}\right] \qquad \begin{matrix}[\because \cos 0 = 1; \\ \cos\pi = -1 \\ \cos 3\pi = -1]\end{matrix}$$

$$= \frac{2\omega l}{\pi}\left[a_1 + \frac{a_2}{3}\right]$$

$$H = \frac{2\omega l}{\pi}\left[a_1 + \frac{a_2}{3}\right] \qquad \ldots(10)$$

Substitute (9) and (10) values in equation (2).

$$(2) \Rightarrow \quad \pi = U - H$$

$$\Rightarrow \quad \pi = \frac{EI}{4}\frac{\pi^4}{l^3}\,(a_1^2 + 81a_2^2) - \frac{2\omega l}{\pi}\left[a_1 + \frac{a_2}{3}\right] \qquad \ldots(11)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \; and \; \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \qquad \frac{\partial\pi}{\partial a_1} = \frac{EI}{4}\frac{\pi^4}{l^3}\,(2a_1) - \frac{2\omega l}{\pi} = 0$$

$$\Rightarrow \frac{EI}{4}\frac{\pi^4}{l^3}\,(2a_2) = \frac{2\omega l}{\pi}$$

$$\Rightarrow \qquad a_1 = \frac{4\omega l^4}{EI\pi^5}$$

Similarly, $$\frac{\partial\pi}{\partial a_2} = \frac{EI}{4}\frac{\pi^4}{l^3}\,(162a_2) - \frac{2\omega l}{\pi}\left(\frac{1}{3}\right) = 0$$

$$\Rightarrow \frac{EI}{4}\frac{\pi^4}{l^3}\,(162a_2) = \frac{2\omega l}{\pi} \times \frac{4l^3}{162\,EI\pi^4} = \frac{4\omega l^4}{243\,EI\pi^5}$$

$$\Rightarrow \qquad a_2 = \frac{2\omega l}{3\pi} \times \frac{4l^3}{162\,EI\pi^4} = \frac{4\omega l^4}{243\,EI\pi^5}$$

$$a_2 = \frac{4\omega l^4}{243\,EI\pi^5}$$

We know that,

$$y = a_1 sin\frac{\pi x}{l} + a_2 sin\frac{3\pi x}{l}$$

Substituting a_1, and a_2 values,

$$\Rightarrow \qquad y = \frac{4\omega l^4}{EI\pi^5}\sin\frac{\pi x}{l} + \frac{4\omega l^4}{243\ EI\pi^5} sin\frac{3\pi x}{l} \qquad \text{... (12)}$$

We know that, maximum deflection occurs at x = $\frac{l}{2}$.

Substitute in x = $\frac{l}{2}$ equation (12),

$$\Rightarrow \qquad y_{max} = \frac{4\omega l^4}{EI\pi^5}\sin\frac{\pi \times \frac{l}{2}}{l} + \frac{4\omega l^4}{243\ EI\pi^5} sin\frac{3\pi \times \frac{l}{2}}{l}$$

$$\Rightarrow \qquad y_{max} = \frac{4\omega l^4}{EI\pi^5}\sin\frac{\pi}{2} + \frac{4\omega l^4}{243\ EI\pi^5} sin\frac{3\pi}{2}$$

$$y_{max} = \frac{4\omega l^4}{EI\pi^5} - \frac{4\omega l^4}{243\ EI\pi^5}$$

$$= \frac{4\omega l^4}{EI\pi^5}\left[1 - \frac{1}{243}\right] \qquad \left[\because \sin\frac{\pi}{2} = 1; \sin\frac{3\pi}{2} = -1\right]$$

$$= \frac{4\omega l^4}{EI\pi^5}(0.9958) = \frac{3.98\omega l^4}{EI\ \pi^5}$$

$$\Rightarrow \qquad y_{max} = 0.0130\frac{\omega l^4}{EI} \qquad \text{... (13)}$$

We know that, simply supported beam subjected to uniformly distributed load, maximum deflection is,

$$y_{max} = \frac{5}{384}\frac{\omega l^4}{EI}$$

$$y_{max} = 0.0130\frac{\omega l^4}{EI} \qquad \text{... (14)}$$

From equations (13) and (14), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are same.

Bending moment at Mid span

We know that,

Bending moment, $$\mathrm{M} = \mathrm{EI}\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} \qquad \ldots (15)$$

From equation (4), we know

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = -\left[\frac{a_1\pi^2}{l^2} sin\left(\frac{\pi x}{l}\right) + \frac{a_2 9\pi^2}{l}\sin\frac{3\pi x}{l}\right]$$

Substituting a_1, and a_2 values,

$$\Rightarrow \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = -\left[\frac{4\omega l^4}{EI\,\pi^4} \times \frac{\pi^2}{l^2} sin\frac{\pi x}{l} + \frac{4\omega l^4}{243\,EI\,\pi^5} \times \frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Maximum bending occurs at x = $\frac{l}{2}$.

$$\Rightarrow \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = -\left[\frac{4\omega l^4}{EI\,\pi^4} \times \frac{\pi^2}{l^2} sin\frac{\pi\frac{l}{2}}{l} + \frac{4\omega l^4}{243\,EI\,\pi^5} \times \frac{9\pi^2}{l^2}\sin\frac{3\pi\frac{l}{2}}{l}\right]$$

$$= -\left[\frac{4\omega l^4}{EI\,\pi^4} \times \frac{\pi^2}{l^2} sin\frac{\pi}{2} + \frac{4\omega l^4}{243\,EI\,\pi^5} \times \frac{9\pi^2}{l^2}\sin\frac{3\pi}{2}\right]$$

$$= -\left[\frac{4\omega l^4}{EI\,\pi^4}\frac{\pi^2}{l^2}(1) + \frac{4\omega l^4}{243\,EI\,\pi^5} \times \frac{9\pi^2}{l^2}(-1)\right]$$

$$\left[\because \sin\frac{\pi}{2} = 1; \sin\frac{3\pi}{2} = -1\right]$$

$$= -\left[\frac{4\omega l^2}{EI\,\pi^5} - \frac{36\omega l^2\pi^2}{243\,EI\,\pi^5}\right]$$

$$= -\frac{4\omega l^2}{EI\,\pi^3} - \frac{36\omega l^2}{243\,EI\,\pi^3}$$

$$= -\frac{4\omega l^2}{EI\,\pi^3} + \frac{0.148\omega l^2}{EI\,\pi^3} = -3.852\frac{\omega l^2}{EI\,\pi^3}$$

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = -0.124\frac{\omega l^2}{EI}$$

Substituting $\frac{d^2y}{dx^2}$ value in bending moment equation,

$$(15) \Rightarrow \quad M_{centre} = EI \times (-0.124)\frac{\omega l^2}{EI}$$

$$\Rightarrow \quad M_{centre} = -0.124\omega l^2 \qquad \text{... (16)}$$

[Negative sign indicates downward load]

We know that, for simply supported beam subjected to uniformly distributed load, maximum bending moment is,

$$M_{centre} = \frac{\omega l^2}{8}$$

$$M_{centre} = 0.125\,\omega l^2 \qquad \text{... (17)}$$

From equation (16) and (17), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are almost same. In order to get accurate result, more terms in Fourier series should be taken.

Example: 1.26

A beam AB of span '*l*' simply supported at ends and carrying a concentrated load W at the centre 'C' as shown in Fig. Determine the deflection at midspan by using Rayleigh-Ritz method and compare with exact solutions.

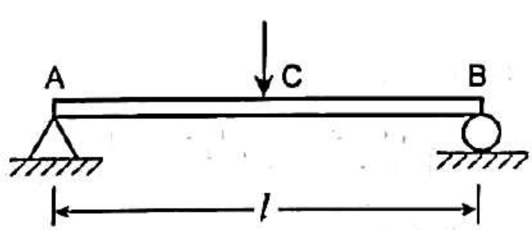
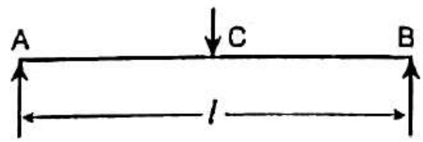


Fig. (i)

Given:



To find: Deflection at midspan, y_{max}.

Solution: From example (1.24). we know that,

$$\text{Deflection, } y = a_1 sin\frac{\pi x}{l} + a_2 sin\frac{3\pi x}{l} \qquad \text{... (1)}$$

Total potential energy of the beam,

$$\pi = U - H \qquad \text{...(2)}$$

Where, U → Strain energy.

H → Work done by external force.

The strain energy, U, of the beam due to bending is given by,

$$U = \frac{EI}{2}\int_0^1 \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \text{... (3)}$$

From equation (9) in previous example problem (1.18), we know that,

$$U = \frac{EI}{4}\frac{\pi^4}{l^3}[a_1^2 + 81a_2^2] \qquad \text{... (4)}$$

$$\text{Work done by external force, H} = W\, y_{max} \qquad \text{...(5)}$$

We know,

$$\text{Deflection,} \qquad y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

In the span, deflection is maximum at x = $\frac{l}{2}$

$$\Rightarrow \qquad y_{max} = a_1 \sin\frac{\pi \times \frac{l}{2}}{l} + a_2 sin\frac{3\pi \times \frac{l}{2}}{l}$$

$$= a_1 \sin\frac{\pi}{2} + a_2 sin\frac{3\pi}{2}$$

$$y_{max} = a_1 - a_2 \qquad \ldots(6)$$

$$\left[\because \sin\frac{\pi}{2} = 1; \sin\frac{3\pi}{2} = -1\right]$$

Substitute y_{max} value in equation (5),

$$\Rightarrow \qquad H = W(a_1 - a_2) \qquad \ldots(7)$$

Substitute U and H values in equation (2),

$$\Rightarrow \qquad \pi = \frac{EI}{4}\frac{\pi^4}{l^3}\,(a_1^2 + 81a_2^2) - W(a_1 - a_2) \qquad \ldots(8)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \; and \; \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \qquad \frac{\partial\pi}{\partial a_1} = \frac{EI}{4}\frac{\pi^4}{l^3}\,(2a_1) - W = 0$$

$$\Rightarrow \frac{EI}{4}\frac{\pi^4}{l^3}\,(2a_2) - W = 0$$

$$\Rightarrow \qquad \frac{EI}{2}\frac{\pi^4}{l^3}\,(a_1) = W$$

$$\Rightarrow \qquad a_1 = \frac{2l^3W}{EI\pi^4}$$

Similarly, $$\frac{\partial\pi}{\partial a_2} = \frac{EI}{4}\frac{\pi^4}{l^3}\,(162a_2) + W = 0$$

$$\Rightarrow \frac{EI}{4}\frac{\pi^4}{l^3}\,(162a_2) + W = 0$$

$$\Rightarrow \qquad \frac{81EI}{2}\frac{\pi^4}{l^3}\,a_2 = -W$$

$$\Rightarrow \qquad a_2 = \frac{2l^3W}{81EI\pi^4} \qquad \text{... (10)}$$

We know that,

Maximum deflection, $y_{max} = a_1 - a_2$

$$\Rightarrow \qquad y_{max} = \frac{2l^3W}{EI\pi^4} + \frac{2l^3W}{81EI\pi^4} = \frac{2l^3W}{EI\pi^4}\left(1 + \frac{1}{81}\right)$$

$$= \frac{2l^3W}{EI\pi^4}(1.0123)$$

$$= \frac{2.0246\, l^3W}{EI\pi^4} = 0.0207\frac{Wl^3}{EI}$$

$$\Rightarrow \qquad y_{max} = \frac{Wl^4}{48.1EI} \qquad \text{... (11)}$$

We know that, simply supported beam subjected to point load at centre, maximum deflection is,

$$y_{max} = \frac{Wl^3}{48EI} \qquad \text{... (12)}$$

From equations (11) and (12), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are same. In order to get accurate result, more terms in Fourier series should be taken.

Example 1.27

A simply supported beam subjected to uniformly distributed load over entire span and it is subjected to a point load at the centre of the span. Calculate the bending moment and deflection at midspan by using Rayleigh-Ritz method and compare with exact solution.

Given:

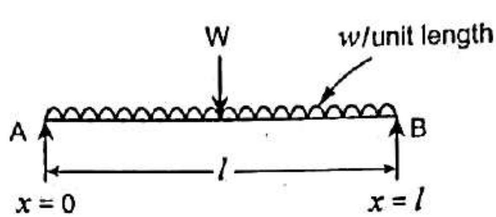


Fig. (i)

To find: 1. Deflection and Bending moment at midspan.

2. Compare with exact solutions.

Solution: From Example 1.24, we know that,

$$\text{Deflection, } y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \qquad \text{... (1)}$$

Total potential energy of the beam is given by,

$$\pi = U - H \qquad \text{...(2)}$$

Total strain energy U of the beam due to bending is given by,

$$U = \frac{EI}{2}\int_0^1 \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \text{... (3)}$$

From equation (9) in Example 1.18, we know that,

$$U = \frac{EI}{2}\frac{\pi^4}{l^4}[a_1^2 + 81a_2^2] \qquad \text{... (4)}$$

Work done by external force,

$$H = \int_0^l \omega\, y\, dx + W\, y_{max} \qquad \text{... (5)}$$

From equation (10) in Example 1.24, we know that,

$$\int_0^l \omega\, y\, dx = \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) \qquad \text{... (6)}$$

We know that,

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

In the span, deflection is maximum at x = $\frac{l}{2}$

$$\Rightarrow \qquad y_{max} = a_1 \sin\frac{\pi \times \frac{l}{2}}{l} + a_2 sin\frac{3\pi \times \frac{l}{2}}{l}$$

$$= a_1 \sin\frac{\pi}{2} + a_2 sin\frac{3\pi}{2}$$

$$y_{max} = a_1 - a_2 \qquad \ldots(7)$$

$$\left[\because \sin\frac{\pi}{2} = 1; \sin\frac{3\pi}{2} = -1\right]$$

$$(5) \Rightarrow \qquad H = \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) + W(a_1 - a_2) \qquad \ldots(8)$$

Substituting U and H values in equation (2) get

$$\Rightarrow \qquad \pi = \frac{EI}{4}\frac{\pi^4}{l^3}\,(a_1^2 + 81a_2^2) - \left[\frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) + W(a_1 - a_2)\right]$$

$$\Rightarrow \qquad \pi = \frac{EI}{4}\frac{\pi^4}{l^3}\,(a_1^2 + 81a_2^2) - \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) - W(a_1 - a_2) \ldots(9)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \text{ and } \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \qquad \frac{\partial\pi}{\partial a_1} = \frac{EI}{4}\frac{\pi^4}{l^3}\,(2a_1) - \frac{2\omega l}{\pi} - W = 0$$

$$\Rightarrow \frac{EI}{2}\frac{\pi^4}{l^3}\,a_1 - \frac{2\omega l}{\pi} - W = 0$$

$$\Rightarrow \qquad \frac{EI}{2}\frac{\pi^4}{l^3}\,(a_1) = \frac{2\omega l}{\pi} + W$$

$$\Rightarrow \qquad a_1 = \frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi} + W\right) \qquad \ldots(10)$$

Similarly, $$\frac{\partial\pi}{\partial a_2} = \frac{EI}{4}\frac{\pi^4}{l^3}\,(162a_2) - \frac{2\omega l}{\pi}\left(\frac{1}{3}\right) + W = 0$$

$$\Rightarrow \frac{EI}{4}\frac{\pi^4}{l^3}\,(162a_2) - \frac{2\omega l}{3\pi} + W = 0$$

$$\Rightarrow \qquad \frac{EI}{4}\frac{\pi^4}{l^3}(162a_2) = \frac{2\omega l}{3\pi} - W$$

$$\Rightarrow \qquad a_2 = \frac{4l^3}{162EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)$$

$$\Rightarrow \qquad a_2 = \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \qquad \ldots (11)$$

From equation (7),

We know that,

Maximum deflection, $y_{max} = a_1 - a_2$

$$\Rightarrow \qquad y_{max} = \frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) - \frac{2l^3 W}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)$$

$$\Rightarrow \qquad y_{max} = \frac{4\omega l^3}{EI\pi^5}\left(1 - \frac{1}{243}\right) - \frac{2Wl^3}{81EI\pi^4}\left(1 + \frac{1}{81}\right)$$

$$= \frac{3.98\omega l^3}{EI\pi^5} - \frac{202Wl^3}{81EI\pi^4}$$

$$= \left[0.0130\frac{l^3\omega}{EI\pi^4} + 0.0207\frac{Wl^3}{EI}\right]$$

$$\Rightarrow \qquad y_{max} = \left[0.0130\frac{l^3\omega}{EI\pi^4} + 0.0207\frac{Wl^3}{EI}\right] \qquad \ldots (12)$$

We know that, simply supported beam subjected to uniformly distributed load, maximum deflection is,

$$y_{max} = \frac{5}{384}\frac{\omega l^3}{EI}$$

Simply support beam subjected to point load at centre, maximum deflection is,

$$y_{max} = \frac{Wl^3}{48EI}$$

So, Total deflection

$$y_{max} = \frac{5}{384}\frac{\omega l^3}{EI} + \frac{Wl^3}{48EI}$$

$$y_{max} = 0.0130\frac{\omega l^3}{EI} + 0.0208\frac{Wl^3}{EI} \qquad \text{... (13)}$$

From equations (12) and (13), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are same.

Bending moment at Mid span

We know that,

Bending moment, $\quad M = EI\frac{d^2y}{dx^2} \qquad \text{... (14)}$

From equation (4), we know

$$\frac{d^2y}{dx^2} = -\left[\frac{a_1\pi^2}{l^2} sin\left(\frac{\pi x}{l}\right) + \frac{a_2 9\pi^2}{l}\sin\frac{3\pi x}{l}\right]$$

Substituting a_1, and a_2 values from equation (10) and (11)

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{2l^3}{EI\,\pi^4}\left(\frac{2\omega l}{3\pi} + W\right) \times \frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{2l^3}{81EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \times \frac{9\pi^2}{l^2} \times \sin\frac{3\pi x}{l}\right]$$

Maximum bending occurs at x = $\frac{l}{2}$.

$$= -\left[\frac{2l^3}{EI\,\pi^4}\left(\frac{2\omega l}{3\pi} + W\right) \times \frac{\pi^2}{l^2}\sin\frac{\pi\frac{l}{2}}{l} + \frac{2l^3}{81EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \times \frac{9\pi^2}{l^2} \times \sin\frac{3\pi\frac{l}{2}}{l}\right]$$

$$= -\left[\frac{2l^3}{EI\,\pi^4}\left(\frac{2\omega l}{3\pi} + W\right) \times \frac{\pi^2}{l^2}(1) + \frac{2l^3}{81EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \times \frac{9\pi^2}{l^2}(-1)\right]$$

$$\left[\because \sin\frac{\pi}{2} = 1; \sin\frac{3\pi}{2} = -1\right]$$

$$= -\left[\frac{2l}{EI\,\pi^2}\left(\frac{2\omega l}{3\pi} + W\right) - \frac{2l}{9\pi^2 EI}\left(\frac{2\omega l}{3\pi} - W\right)\right]$$

$$= -\left[\frac{4\omega l^2}{EI\,\pi^3} + \frac{2\omega l}{EI\,\pi^2} - \frac{4\omega l^2}{27\,\pi^3 EI} + \frac{2\omega l}{9\pi^2 EI}\right]$$

$$= -\left[\frac{4\omega l^2}{EI\,\pi^3}\left(1 - \frac{1}{27}\right) + \frac{2\omega l}{EI\,\pi^2}\left(1 + \frac{1}{9}\right)\right]$$

$$= -\left[\frac{3.851\omega l^2}{EI\,\pi^3} + 2.222\frac{Wl}{EI\,\pi^2}\right]$$

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = -\left[0.124\,\frac{\omega l^2}{EI} + 0.225\frac{Wl}{EI}\right]$$

Substituting $\frac{d^2y}{dx^2}$ value in bending moment equation,

$$(14) \Rightarrow \qquad M_{centre} = EI\frac{d^2y}{dx^2} = -EI\left[0.124\,\frac{\omega l^2}{EI} + 0.225\frac{Wl}{EI}\right]$$

$$\Rightarrow \qquad M_{centre} = -(0.124\omega l^2 + 0.225Wl) \qquad \ldots(15)$$

[Note: Negative sign indicates downward load]

We know that, for simply supported beam subjected to uniformly distributed load, maximum bending moment is,

$$M_{centre} = \frac{\omega l^2}{8}$$

Simply supported beam subjected to point load at centre, maximum bending moment is,

$$M_{centre} = \frac{Wl}{4}$$

Total bending moment,

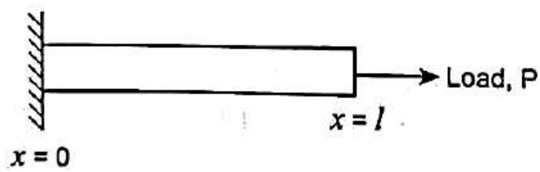
$$M_{centre} = \frac{\omega l^2}{8} + \frac{Wl}{4}$$

$$M_{centre} = 0.125\omega l^2 + 0.25\, Wl \qquad \text{... (16)}$$

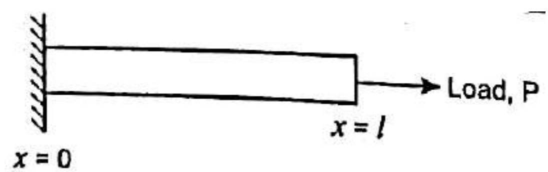
From equation (15) and (16), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are almost same. In order to get accurate result, more terms in Fourier series should be taken.

Example 1.28

A bar uniform cross section is clamped at one end and left free at the other end and it is subjected to a uniform axial load P as shown in Fig. Calculate the displacement and stress in a bar by using two terms polynomial and three terms polynomial. Compare with exact solutions.



Given:



To find: 1. Displacement of the bar, δu.

2. stress in the bar, σ

By using two terms and three terms polynomial

Solution: We know that, Polynomial function for displacement is,

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_n x^n$$

Case (i): Considering two terms of polynomial,

$$i.e.,\ \ u = a_0 + a_1 x \qquad \text{...(1)}$$

Apply boundary condition,

at x = 0, u = 0

$\Rightarrow \quad 0 = a_0 + 0$

$\Rightarrow \quad a_0 = 0$

Substituting a_0 value in equation (1),

$$\Rightarrow \quad u = a_1 x \qquad \ldots(2)$$

$$\frac{du}{dx} = a_1$$

We know that,

Total potential energy of the beam, $\pi = U - H$ …(3)

Where, U → Strain energy of the bar.

H → Work done by external force of the bar.

$$\text{Strain energy, } U = \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{EAa_1^2}{2}[x]_0^l$$

$$U = \frac{EAa_1^2 l}{2} \qquad \ldots(4)$$

Work done by external force,

$$H = \int_0^l P\,dx = \int_0^l \rho\, u\, A\, dx \qquad [\because Load, P = \rho\, u\, A]$$

$$= \rho\, A\int_0^l u\,dx = \rho\, A\int_0^l a_1\, x\, dx \qquad [\because u = a_1 x]$$

$$= \rho\, Aa_1\left[\frac{x^2}{2}\right]_0^l = \frac{\rho\, A\, a_1}{2}\,[l^2]$$

$$H = \frac{\rho\, A\, a_1 l^2}{2} \qquad \ldots (5)$$

Substitute U and H values in equation (3)

(3) $\Rightarrow$ $$\pi = \frac{EA\, A\, a_1^2 l}{2} - \frac{\rho\, A\, a_1 l^2}{2}$$

For stationary value of π, the following condition must be satisfied.

$$i.e., \qquad \frac{\partial \pi}{\partial a_1} = 0$$

$$\Rightarrow \frac{EA\,(2a_1)l}{2} - \frac{\rho\, A\, l^2}{2} = 0$$

$$\Rightarrow \quad EA\, a_1 l - \frac{\rho\, A\, l^2}{2} = 0$$

$$\Rightarrow \qquad EA\, a_1 l = \frac{\rho\, A\, l^2}{2}$$

$$\Rightarrow \qquad a_1 = \frac{\rho\, l}{2E}$$

Substitute a_1 value in equation (2),

$$\Rightarrow \qquad u = a_1 x = \frac{\rho\, l}{2E} \times x$$

$$u = \frac{\rho\, l}{2E} \times x$$

We know that, Extension of the bar,

$$\delta u = u_1 - u_0 = \frac{\rho\, l}{2E} \times l - 0$$

$$[\because At\ x = l, u = u_1, At\ x = 0, u = u_0 = 0]$$

$$= \frac{\rho\, l^2}{2E}$$

Extension or displacement of the bar,

$$\delta u = \frac{\rho\, l^2}{2E} \qquad \ldots (6)$$

Stress in the bar,

$$\sigma = E\frac{du}{dx} = E \times \frac{\rho\, l}{2E} \qquad \left[\because u = \frac{\rho\, l}{2E} \times x\right]$$

$$\sigma = \frac{\rho\, l}{2} \qquad \text{... (7)}$$

Case (ii): Considering three terms of polynomial,

$$i.e., u = a_0 + a_1 x + a_2 x^2 \qquad \text{... (8)}$$

Apply boundary condition, at $x = 0$, u = 0

$\Rightarrow \quad 0 = a_0 + 0 + 0$

$\Rightarrow \quad a_0 = 0$

Substituting a_0 value in equation (1),

$$(8) \Rightarrow \quad u = a_1 x + a_2 x^2 \qquad \text{...(9)}$$

$$\frac{du}{dx} = a_1 + 2a_2 x$$

We know that,

Total potential energy of the beam, π = U – H ...(10)

$$\text{Strain energy}, U = \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{EA}{2}\int_0^l (a_1 + 2a_2 x)^2 dx$$

$$= \frac{EA}{2}\int_0^l [a_1^2 + (2a_2 x)^2 + 2a_1 2a_2 x]\, dx$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \frac{EA}{2}\int_0^l [a_1^2 dx + 4a_2 x^2 dx + 4a_1 2a_2 x\, dx]$$

$$= \frac{EA}{2}\left[a_1^2(x)_0^l + 4a_2^2\left(\frac{x^3}{3}\right)_0^l + 4a_1a_2\left(\frac{x^2}{3}\right)_0^l\right]$$

$$= \frac{EA}{2}[a_1^2(l-0) + \frac{4a_2^2}{3}(l^3-0) + \frac{4a_1a_2}{2}(l^2-0)]$$

$$U = \frac{EA}{2}\left[a_1^2 l + \frac{4a_2^2}{3}(l^3) + 2a_1a_2(l^2)\right] \quad \text{... (4)}$$

Work done by external force,

$$H = \int_0^l P\,dx = \int_0^l \rho\,u\,A\,dx \qquad [\because Load, P = \rho\,u\,A]$$

$$= \rho\,A\int_0^l u\,dx = \rho\,A\int_0^l a_1x + a_2x^2\,dx$$

$$= \rho\,A\int_0^l [a_1x\,dx + a_2x^2dx]$$

$$= \rho\,A\left[a_1\left[\frac{x^2}{2}\right]_0^l + a_2\left[\frac{x^3}{3}\right]_0^l\right]$$

$$= \rho\,A\left[\frac{a_1}{2}\,[l^2-0] + \frac{a_2}{3}\,[l^3-0]\right]$$

$$H = \rho\,A\left[\frac{a_1}{2}l^2 + \frac{a_2}{3}l^3\right] \quad \text{... (12)}$$

Substitute (11) and (12) values in (10),

$(10) \Rightarrow \qquad \pi = U - H$

$$\pi = \frac{EA}{2}\left[a_1^2 l + \frac{4a_2^2}{3}(l^3) + 2a_1a_2(l^2)\right] - \rho\,A\left[\frac{a_1}{2}l^2 + \frac{a_2}{3}l^3\right] \quad \text{... (13)}$$

For stationary value of π, the following condition must be satisfied.

$$i.e., \quad \frac{\partial \pi}{\partial a_1} = 0 \; and \; \frac{\partial \pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{\partial \pi}{\partial a_1} = \frac{EA}{2}\left[a_1^2 l + \frac{4a_2^2}{3}(l^3) + 2a_1 a_2 (l^2)\right] - \rho\, A\left[\frac{a_1}{2} l^2 + \frac{a_2}{3} l^3\right] = 0$$

$$\Rightarrow \quad \frac{EA}{2}[2a_1 l + 2a_2(l^2)] - \rho\, A\left[\frac{l^2}{2}\right] = 0$$

$$\Rightarrow \quad EA\,[a_1 l + a_2 l^2] - \frac{\rho\, A\, a_1 l^2}{2} = 0$$

$$\Rightarrow \quad a_1 + a_2 l = \frac{\rho\, l}{2E} \qquad \ldots (14)$$

$$\text{Similarly,} \quad \frac{\partial \pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{EA}{2}\left[0 + \frac{8a_2}{3}(l^3) + 2a_1(l^2)\right] - \rho\, A\left[0 + \frac{l^3}{3}\right] = 0$$

$$\Rightarrow \quad \frac{EA}{2}\left[\frac{8}{3}a_2(l^3) + 2a_1(l^2)\right] = \frac{\rho\, Al^3}{3}$$

$$\Rightarrow \quad \frac{8}{3}a_2(l^3) + 2a_1(l^2) = \frac{2\rho\, Al^3}{3EA}$$

$$\Rightarrow \quad \frac{8}{3}a_2(l^3) + 2a_1(l^2) = \frac{2\rho\, l^3}{3E}$$

$$\Rightarrow \quad \frac{4}{3}a_2(l^3) + a_1(l^2) = \frac{\rho\, l^3}{3E}$$

$$\Rightarrow \quad a_1 + \frac{4}{3}a_2 l = \frac{\rho\, l}{3E}$$

$$\Rightarrow \quad a_1 + 1.333 a_2 l = \frac{\rho\, l}{3E} \qquad \ldots (15)$$

Solving (14) and (15),

$$a_1 + a_2 l = \frac{\rho\, l}{2E}$$

$$a_1 + 1.333\; a_2 l = \frac{\rho\, l}{2E}$$

$$a_2 l - 1.333\; a_2 l = \frac{\rho\, l}{2E} - \frac{\rho\, l}{3E}$$

$$-0.333\; a_2 l = \frac{\rho\, l}{E}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\rho\, l}{E}\left(\frac{3-2}{6}\right)$$

$$-0.333\; a_2 l = \frac{\rho\, l}{6E}$$

$$\Rightarrow \qquad -0.333\; a_2 = \frac{\rho}{6E}$$

$$\Rightarrow \qquad a_2 = \frac{-\rho}{2E}$$

Substituting a_2 value in equation (14)

$$(14) \qquad \Rightarrow \qquad a_1 + \left(\frac{-\rho}{2E}\, l\right) = \frac{\rho\, l}{2E}$$

$$a_1 - \frac{\rho l}{2E} = \frac{\rho\, l}{2E}$$

$$a_1 = \frac{\rho l}{2E} + \frac{\rho\, l}{2E}$$

$$a_1 = \frac{\rho l}{E}$$

We know that, $\qquad u = a_1 x + a_2 x^2$

Substitute a_1 and a_2 values,

$$u = \frac{\rho l}{E} x - \frac{\rho}{E} x^2$$

$$u = \frac{\rho}{E}\left[lx - \frac{x^2}{2}\right] \qquad \text{... (16)}$$

At x = l, u = u_1 substitute in equation (16),

$$\Rightarrow \qquad u_1 = \frac{\rho}{E}\left[l^2 - \frac{l^2}{2}\right]$$

$$u_1 = \frac{\rho}{E} \times \frac{l^2}{2}$$

We know that, Extension of the bar,

$$\delta u = u_1 - u_0 = \frac{\rho\, l^2}{2E} - 0$$

$$[\because At\ x = 0, u = u_0 = 0]$$

$$= \frac{\rho\, l^2}{2E}$$

Extension or displacement of the bar, $\delta u = \frac{\rho\, l^2}{2E}$ ... (17)

From equation (16), we know that,

$$u = \frac{\rho}{E}\left[lx - \frac{x^2}{2}\right]$$

$$\frac{du}{dx} = \frac{\rho}{E}\left[lx - \frac{2x}{2}\right] = \frac{\rho}{E}\ (l - x)$$

Stress in the bar,

$$\sigma = E\frac{du}{dx} = E \times \frac{\rho}{E}(l - x)$$

$$\sigma = \rho(l - x) \qquad \text{... (18)}$$

Exact Solution: We know that, actual extension of the bar,

$$\delta l = \int_0^l \frac{Pdx}{AE} = \int_0^l \frac{PAx}{AE}dx \qquad [\because\ P = \rho\, Ax]$$

$$= \frac{\rho}{E}\int_{0}^{l} x\,dx = \frac{\rho}{E}\left[\frac{x^2}{2}\right]_0^l = \frac{\rho}{E}\left[\frac{l^2}{2}\right]$$

$$\delta l = \frac{\rho l^2}{2E}$$

From equation (6), (17) and (19), we know that total extension of the bar obtained is exact in both the cases.

Result:

1. Displacement of the bar,

$$\delta u = \frac{\rho l^2}{2E} \qquad \text{[Two terms polynomial]}$$

$$\delta u = \frac{\rho l^2}{2E} \qquad \text{[Three terms polynomial]}$$

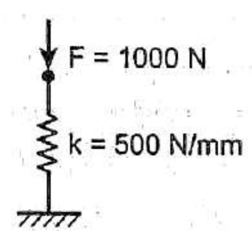
2. Stress in the bar,

$$\sigma = \frac{\rho l}{2} \qquad \text{[Two terms polynomial]}$$

$$\sigma = \rho(l - x) \qquad \text{[Three terms polynomial]}$$

Example 1.29

A linear elastic spring is subjected to a force of 1000 N as shown in Fig. Calculate the displacement and the potential energy of the spring.



Given: Force, F = 1000 N

Stiffness, k = 500 N/mm

To find: 1. Displacement, *x*.

2. Potential energy, π.

Solution: we know that,

Total potential energy of the beam, $\pi = U - H$ …(1)

Where, U = Strain energy $= \frac{l}{2}(kx) \times x$

H = Work done by external force = Fx

Substitute U and H values in equation (1)

$$\Rightarrow \quad \pi = \frac{1}{2}(k, x) \times x - Fx$$

$$\pi = \frac{1}{2}kx^2 - Fx \qquad \text{... (2)}$$

For stationary value of π, $\frac{\partial \pi}{\partial \mathrm{x}} = 0$

$$\Rightarrow \quad \frac{1}{2} \times 2kx - F = 0$$

$$\Rightarrow \quad kx - F = 0$$

$$\Rightarrow \quad 500(x) - 1000 = 0$$

$$\Rightarrow \quad 500(x) = 1000$$

$$\Rightarrow \quad x = 2mm$$

Substitute x values in equation (2),

$$(2) \Rightarrow \quad \pi = \frac{1}{2}kx^2 - Fx = \frac{1}{2}(500)(2)^2 - 1000(2)$$

$$\pi = -1000\, N - mm$$

Result:

1. Displacement, x = 2mm
2. Potential energy, π = -1000 N – mm

Example 1.30

Consider a 1 mm diameter, 50 mm long aluminium pin-fin as shown in Fig.(i) used to enhance the heat transfer from a surface wall maintained at 300°C. Calculate the temperature distribution in a pin fin by using Rayleigh-Ritz method, Take, k = 200 w/m°C for aluminium h = 20 w/m°C, $T_\infty = 30^o C$.

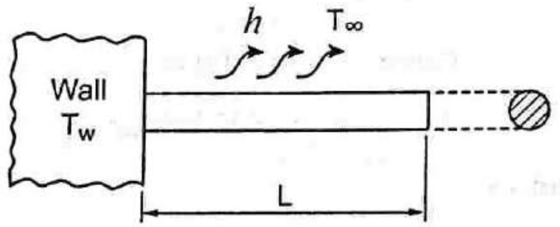


Fig. (i)

$$k\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)$$

$$T(0) = T_w = 300^o C$$

$$q_L = kA\frac{dT}{dx}(L) = 0 \text{ (Insulated tip)}$$

Given: The governing differential equation,

$$k\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)$$

Diameter, d = 1 mm = 1×10^{-3} m

Length, L = 50 mm = 50×10^{-3} m

Thermal conductivity, k = 200 w/m°C

Heat transfer coefficient, h = 20 w/m°C

Fluid temperature, T_∞ = 30°C

Boundary conditions, T (0) = T_W = 300°C

$$q_L = kA\frac{dT}{dx}(L) = 0$$

To find: Ritz parameters.

Solution: The equivalent functional representation is given by

π = Strain Energy – Work done

π = U – W

$$\pi = \left[\int_0^L \frac{1}{2}k\left(\frac{dT}{dx}\right)^2 dx + \int_0^L \frac{1}{2}\frac{Ph}{A}(T - T_\infty)^2 dx\right] - q_L T_L \qquad \text{... (1)}$$

$$\pi = \int_0^L \frac{1}{2}k\left(\frac{dT}{dx}\right)^2 dx + \int_0^L \frac{1}{2}\frac{Ph}{A}(T - T_\infty)^2 dx \qquad \text{... (2)}$$

$[\because q_L = 0]$

Assume a trial function, let T(x) $= a_0 + a_1 x + a_2 x^2$ ... (3)

Apply boundary condition, at $x = 0$, T(x) = 300

$$300 = a_0 + a_1(0) + a_2(0)^2$$

$$a_0 = 300$$

Substituting a_0 value in equation (3),

$$\mathrm{T(x)} = 300 + a_1 x + a_2 x^2 \qquad \text{...(4)}$$

$$\Rightarrow \frac{dT}{dx} = a_1 + 2a_2 x \qquad \text{... (5)}$$

Substitute the equation (4), (5) in (2)

$$\pi = \left[\int_0^L \frac{1}{2}k(a_1 + 2a_2 x)^2 dx + \int_0^L \frac{1}{2}\frac{Ph}{A}(300 + a_1 x + a_2 x^2 - 30)^2 dx\right]$$

$$\pi = \int_0^L \frac{1}{2}k(a_1 + 2a_2 x)^2 dx + \int_0^L \frac{1}{2}\frac{Ph}{A}(270 + a_1 x + a_2 x^2)^2 dx$$

$$\left[\begin{array}{c}\because (a+b)^2 = a^2 + b^2 + 2ab; \\ (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca\end{array}\right]$$

$$\pi = \frac{k}{2}\int_0^L [a_1^2 + 4a_2^2 x + 4a_1 a_2 x]$$

$$+ \frac{Ph}{2A}\int_0^L [(270)^2 + a_1^2 x^2 + a_2^2 x^4 + 540a_1 x + 2a_1 a_2 x^3$$

$$+ 540a_2 x^2)]dx$$

$$\pi = \frac{k}{2}\left[a_1^2 + \frac{4a_2^2 x^3}{3} + \frac{4a_1 a_2 x^2}{2}\right]_0^{50\times10^{-3}}$$

$$+ \frac{Ph}{2A}\int_0^L \left[72900x + \frac{a_1^2 x^3}{3} + \frac{a_2^2 x^5}{5} + \frac{540a_1 x^2}{2}\right.$$

$$\left. + \frac{2a_1 a_2 x^4}{4} + \frac{540a_2 x^3}{3}\right]_0^{50\times10^{-3}}$$

$$\pi = \frac{k}{2}\left[(50\times10^{-3})a_1^2 + \frac{4a_2^2(50\times10^{-3})^3}{3} + \frac{4a_1 a_2 (50\times10^{-3})^2}{2}\right]$$

$$+ \frac{Ph}{2A}\left[72900(50\times10^{-3}) + \frac{a_1^2(50\times10^{-3})^3}{3}\right.$$

$$+ \frac{a_2^2(50\times10^{-3})^5}{5} + \frac{540a_1(50\times10^{-3})^2}{2}$$

$$\left. + \frac{2a_1 a_2(50\times10^{-3})^4}{4} + \frac{540a_2(50\times10^{-3})^3}{3}\right]$$

$$\pi = \frac{200}{2}[50\times10^{-3}a_1^2 + 1.666\times10^{-4}a_2^2 + 50\times10^{-3}a_1 a_2]$$

$$+ \frac{\pi\times10^{-3}\times20}{2\times\frac{\pi}{4}\times(10^{-3})^2} + [3645 + 4.166\times10^{-5}a_1^2$$

$$+ 6.25\times10^{-8}a_2^2 + 0.675a_1 + 3.125\times10^{-6}a_1 a_2$$

$$+ 0.0225a_2]$$

$$\pi = [5a_1^2 + 0.0166a_2^2 + 0.5a_1a_2] + [14.58 \times 10^7 + 1.66a_1^2 + 2.5 \times 10^{-3}a_2^2 + 2700a_1 + 0.125a_1a_2 + 900a_2]$$

$$\pi = [6.66a_1^2 + 0.0191a_2^2 + 0.625a_1a_2 + 27000a_1 + 900a_2 + 14.58 \times 10^7]$$

Apply, $\frac{\partial \pi}{\partial a_1} = 10$

$$\Rightarrow \; 13.32a_1 + 0.625a_2 + 27000 = 0$$

$$\Rightarrow \quad 13.32a_1 + 0.625a_2 = -27000 \quad \ldots (6)$$

Apply, $\frac{\partial \pi}{\partial a_2} = 0$

$$\Rightarrow \; 0.625a_1 + 0.382a_2 + 900 = 0$$

$$\Rightarrow \quad 0.625a_1 + 0.382a_2 = -900 \quad \ldots (7)$$

Solve the equations (6) and (7),

$$13.32a_1 + 0.625a_2 = -27000$$

$$0.625a_1 + 0.382a_2 = -900$$

(6) × 0.625

$$\Rightarrow \quad 8.325a_1 + 0.3906a_2 = -16875$$

(7) × -13.32

$$\Rightarrow \quad -8.325a_1 - 0.50886a_2 = +11988$$

$$-0.1182a_2 = -4887$$

$$a_2 = 41345$$

Substituting a_2 value in equation (6)

$$13.32a_1 + 0.625(41345) = -27000$$

$$\Rightarrow \quad a_1 = 3967.01$$

Substitute a_1 and a_2 values in equation (3)

$$T = 300 - 3967.01x + 41345x^2$$

Result: Temperature distribution in pin-fin

$$T = 300 - 3967.01x + 41345x^2$$

Example 1.31

Using Rayleigh-Ritz method determine the expressions for displacement and stress in fixed bar subjected to axial force P as shown in Fig. (i). Draw the displacement and stress variation diagram. Take three terms in displacement function.

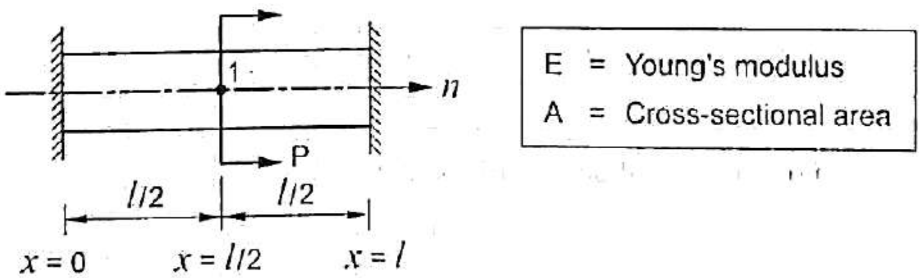


Fig. (i)

Given:

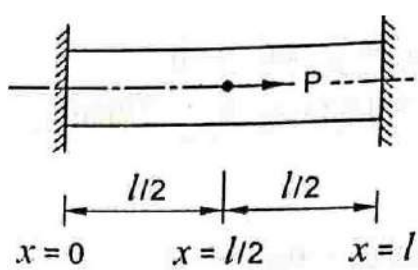


Fig. (ii)

To draw: Displacement and stress variation diagram

Solution: we know that, polynomial function for three terms,

$$i.e., u = a_0 + a_1x + a_2x^2 \quad \ldots(1)$$

This function has to satisfy the boundary condition,

(i) at x = 0, u = 0

(ii) at x = l, u = 0

Apply boundary condition (i), we get

$$a_0 = 0 \qquad \ldots(2)$$

Apply boundary condition (ii), we get

$$0 = a_0 + a_1 l + a_2 l^2 \qquad \ldots(3)$$

$\therefore$ From equations (2) and (3) we get,

$$0 = a_0 + a_1 l + a_2 l^2$$

$$a_1 = -a_2 l \qquad \ldots(4)$$

Substituting a_0 and a_0 values in equation (1),

$$u = 0 - a_1 l\, x + a_2 x^2$$

$$u = a_2[-l\, x + x^2] \qquad \ldots(5)$$

At x $= \frac{l}{2}$,

$$u_1 = a_2\left[-l\,\left(\frac{l}{2}\right) + \left(\frac{l}{2}\right)^2\right]$$

$$u_1 = -\frac{a_2 l^2}{4} \qquad \ldots(6)$$

We know that, Total potential energy of the bar,

$$\pi = U - H$$

Where, U $\rightarrow$ Strain energy of the bar.

H $\rightarrow$ Work done by external force of the bar.

$\therefore$ Potential energy

$$\pi = \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx - p\, u_1 \qquad \ldots(7)$$

We know that, $\quad u = a_2[-l\, x + x^2]$

$$\Rightarrow \quad \frac{du}{dx} = a_2(-l + 2x)$$

Now,
$$\pi = \frac{EA}{2}\int_0^l (a_2(-l+2x))^2 dx - p\left(-\frac{a_2 l^2}{4}\right)$$

$$= \frac{EA}{2}a_2^2\int_0^l (l^2+4x^2-4\,l\,x)dx - p\,a_2\frac{l^2}{4}$$

$$= \frac{EA}{2}a_2^2\left[l^2x+\frac{4x^3}{3}-2\,l\,x^2\right]_0^l + p\,a_2\frac{l^2}{4}$$

$$= \frac{EA}{2}a_2^2\left[l^2+\frac{4l^3}{3}-2\,l\,(l)^2\right]_0^l + p\,a_2\frac{l^2}{4}$$

$$\pi = \frac{EA}{2}a_2^2\left(\frac{l^3}{3}\right) + p\,a_2\frac{l^2}{4}$$

For stationary value of π, the following condition must be satisfied.

$$i.e., \qquad \frac{\partial\pi}{\partial a_2} = 0$$

$$\frac{EA}{2}\left(2\frac{a_2 l^3}{3}\right) + p\frac{l^2}{2} = 0$$

$$EAa_2\frac{l^3}{3} = p\frac{l^2}{4}$$

$$a_2 = -\frac{3p}{4EAl} \qquad \text{... (8)}$$

Substitute a_2 value in equations (5) and (6)

$$u = \frac{-3p}{4EAl}[-l\,x+x^2]$$

$$u_1 = \frac{3pl}{16}$$

Stress in the bar,

$$\sigma = E\frac{du}{dx}$$

$$= Ea_2(-l + 2x)$$

$$= \frac{3Ep}{4Al}(l - 2x)$$

$$\sigma = \frac{3p}{4Al}(l - 2x)$$

We know that,

At $x = 0 \Rightarrow$ $\sigma_0 = \sigma_{x=0} = \frac{3p}{4A}$

At $x = \frac{1}{2} \Rightarrow$ $\sigma_1 = \sigma_{x=\frac{l}{2}} = 0$

At $x = 1 \Rightarrow$ $\sigma_2 = \sigma_{x=l} = -\frac{3p}{4A}$

The variation of displacement and stress diagram are shown in figure.

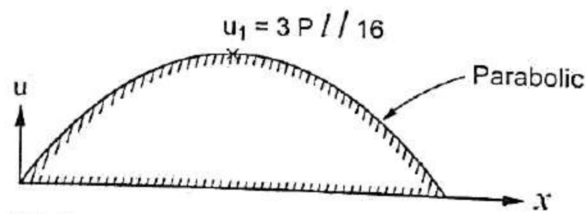


Fig. Variation of displacement

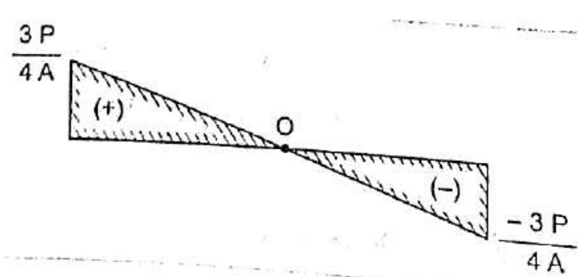


Fig. Variation of stress

Example 1.32

Consider the differential equation for a problem such as $\frac{d^2y}{dx^2} + 300x^2 = 0; 0 \le x \le 1$ with the boundary conditions, y(0) = y(1) = 0, the functional corresponding to this problem to be extremized is given by

$$I = \int_0^l \left\{ -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300x^2 y \right\} dx$$

Find the solution of the problem using Rayleigh Ritz method using a one term solution is $y = a\,x\,(1 - x^3)$.

Given: Differential equation,

$$\frac{d^2y}{dx^2} + 300x^2 = 0; \qquad 0 \le x \le 1$$

Boundary conditions: y(0) = y(1) = 0

(i) x = 0, y = 0

(ii) x = 1, y = 0

$$I = \int_0^l \left\{ -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300x^2 y \right\} dx$$

Trial function, $y = a\,x\,(1 - x^3)$

i.e., $y = a\,x - a\,x^4$ … (1)

It satisfies the boundary conditions,

x = 0, y = 0

x = 1, y = 0

Differential equation,

$$\frac{d^2y}{dx^2} + 300x^2 = 0 \qquad \text{... (2)}$$

$$\Rightarrow \qquad \frac{dy}{dx} = a - 4ax^3$$

$$\left(\frac{dy}{dx}\right)^2 = (a - 4ax^3)^2 \qquad \ldots (3)$$

We know that,
$$I = \int_0^l \left\{-\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300x^2 y\right\} dx \qquad \ldots (4)$$

Substitute the equation (1), (3) in (4)

$$\Rightarrow \qquad I = \int_0^l \left\{-\frac{1}{2}(a - 4ax^3)^2 + 300x^2(a\,x - a\,x^4)\right\} dx$$

$$= \int_0^l \Big[-\frac{1}{2}(a^2 + 16a^2x^6 - 8a^2x^3)$$

$$+ (300a\,x^3 - 300a\,x^6)\Big] dx$$

$$= -\frac{1}{2}\left[a^2x + 16a^2\frac{x^7}{7} - 8a^2\frac{x^4}{4}\right]_0^1$$

$$+ \left[300\frac{ax^4}{4} - 300\frac{ax^7}{7}\right]_0^1$$

$$= -\frac{1}{2}\left[a^2 + \frac{16}{7}a^2 - 2a^2\right] + \frac{300}{4}a - \frac{300}{7}a$$

$$I = -\frac{a^2}{2} - \frac{8}{7}a^2 + a^2 + \frac{300}{4}a - \frac{300}{7}a$$

Apply, $\dfrac{\partial I}{\partial a} = 0$

$$\Rightarrow \qquad -\frac{2a}{2} - \frac{8}{7}(2a) + 2a + \frac{300}{4} - \frac{300}{7} = 0$$

$$-a - \frac{16}{7}a + 2a + \frac{300}{4} - \frac{300}{7} = 0$$

i.e., $$\frac{-16a + 7a}{7} = -\frac{2100 + 1200}{28}$$

i.e., $$-9a = -900 \times \frac{7}{28}$$

$$a = 25$$

Hence the solution is, $y = 25\, x\, (1 - x^3)$

Result: Solution, $y = 25\, x\, (1 - x^3)$

1.19. APPLICATION TO BAR ELEMENT

1.19.1 Bar Element Formulated from the stationarity of a Functional

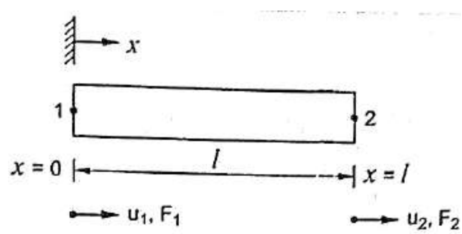


Fig. 1.31 Typical bar Element

Consider a bar element with nodes 1 and 2 as shown in Fig. 1.31. u_1 and u_2 are the displacements at the respective nodes. So, u_1 and u_2 are considered as degrees of freedom of this bar element. [Refer chapter 2, Section 2.6.3, equation no. (2.21)].

Node: Degrees of freedom is nothing but noded displacements.]

$$u = N_1 u_1 + N_2 u_2 \qquad \ldots(1.27)$$

Where, $N_1 = 1 - \frac{x}{l}$

$$N_2 = \frac{x}{l}$$

Substitute the N_1, N_2 values in equation (1.27)

$$u = \left(1 - \frac{x}{l}\right) u_1 + \left(\frac{x}{l}\right) u_2 \qquad \dots (1.28)$$

The strain energy stored within the element is given by,

$$U = \int_0^l \frac{AE}{2} \left(\frac{du}{dx}\right)^2 dx \qquad \dots (1.29)$$

$$U = \frac{AE}{2} \left(\frac{u_2 - u_1}{l}\right)^2 \int_0^l dx$$

$$U = \frac{AE}{2} \left(\frac{u_2 - u_1}{l}\right)^2 (x)_0^l$$

$$U = \frac{AE}{2} \left(\frac{u_2 - u_1}{l}\right)^2 (l) \qquad \dots (1.30)$$

When there is a distributed force q_0 acting at each point on the element and concentrated forces F at the nodes, the potential of the external forces is given by,

$$H = \int_0^l q_0\, u\, dx + F_1\, u_1 + F_2\, u_2$$

$$= q_0 \left(\frac{u_1 + u_2}{2}\right) l\, dx + F_1\, u_1 + F_2\, u_2 \qquad \left[\because u = \frac{u_1 + u_2}{2}\right]$$

$$H = q_0 \frac{l}{2}\, dx + F_1\, u_1 + F_2\, u_2 \qquad \dots (1.31)$$

Thus the total potential energy,

$$\pi = U - H$$

$$\pi = \frac{AE}{2} \left(\frac{u_2 - u_1}{l}\right)^2 - q_0 \frac{l}{2} (u_1 + u_2)\, dx - F_1\, u_1 - F_2\, u_2 \qquad \dots (1.32)$$

Apply, $\dfrac{\partial \pi}{\partial u_1} = 0$

$$\Rightarrow -\frac{AE}{2l}\times 2(u_2-u_1)-\frac{q_0 l}{2}-F_1=0$$

$$\frac{AE}{l}(u_1-u_2)-\frac{q_0 l}{2}-F_1=0$$

$$\Rightarrow \qquad \frac{AE}{l}(u_1-u_2)=\frac{q_0 l}{2}+F_1 \qquad \text{... (1.33)}$$

Similarly, $\dfrac{\partial \pi}{\partial u_2}=0$

$$\Rightarrow \quad \frac{AE}{2l}\times 2(u_2-u_1)-\frac{q_0 l}{2}-F_2=0$$

$$\frac{AE}{l}(u_2-u_1)=\frac{q_0 l}{2}+F_2 \qquad \text{... (1.34)}$$

Arrange the equation (1.33) and 1.34) in matrix form, we get,

$$\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}\dfrac{q_0 l}{2}\\ \dfrac{q_0 l}{2}\end{Bmatrix}+\begin{Bmatrix}F_1\\ F_2\end{Bmatrix} \qquad \text{... (1.35)}$$

[k] {u} = {F}

1.19.2 One – dimensional Heat Transfer Elements Based on the stationary of a Functional

Consider a bar element with nodes 1 and 2 as shown in Fig. 1.32. T_1 and T_2 are the temperatures at the respective nodes. So, T_1 and T_2 considered as degrees if freedom of this bar element.

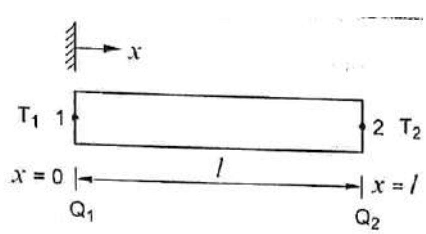


Fig. 1.32. Heat transfer element

We know that, $$T(x) = N_1 T_1 + N_2 T_2 \qquad \ldots(1.36)$$

$$T(x) = \left(1 - \frac{x}{l}\right) T_1 + \left(\frac{x}{l}\right) T_2 \qquad \ldots(1.37)$$

$$\left[\because N_1 = 1 - \frac{x}{l}; N_2 = \frac{x}{l}\right]$$

The strain energy stored within the element is given by,

$$U = \frac{1}{2}\int_0^l k\left(\frac{dT}{dx}\right)^2 dx \qquad \ldots(1.38)$$

Potential energy of external force is given by

$$H = \int_0^l q_0\, T\, dx + Q_1\, T_1 + Q_2\, T_2 \qquad \ldots(1.39)$$

The total potential energy,

$$\pi = U - H$$

$$= \frac{1}{2}\int_0^l k\left(\frac{dT}{dx}\right)^2 dx - \int_0^l q_0\, T\, dx - Q_1\, T_1 - Q_2\, T_2 \qquad \ldots(1.40)$$

From equation (1.37),

$$\frac{dT}{dx} = -\frac{1}{l} T_1 + \frac{1}{l} T_2$$

$$\frac{dT}{dx} = \frac{1}{l}(T_2 - T_1) \qquad \ldots(1.41)$$

Substitute the equation (1.41) in equation (1.40),

$$\pi = \frac{1}{2}\int_0^l k\left(\frac{1}{l}(T_2 - T_1)\right)^2 dx - \int_0^l q_0\, T\, dx - Q\, T_1 - Q\, T_2$$

$$\pi = \frac{k}{2l}((T_2 - T_1)^2 - \frac{q_0 l}{2}(T_1 + T_2) - Q\, T_1 - Q\, T_2$$

We know that,

Apply, $\dfrac{\partial \pi}{\partial T_1} = 0 \Rightarrow$

$$\frac{k}{2l} \times 2(T_2 - T_1)(-1) - \frac{q_0 l}{2} - Q\, T_1 = 0$$

$$\frac{k}{l}(T_1 - T_2) = \frac{q_0 l}{2} + Q\, T_1 \qquad \text{... (1.42)}$$

Similarly, $\dfrac{\partial \pi}{\partial T_2} = 0 \Rightarrow$

$$\frac{k}{2l} \times 2(T_2 - T_1) - \frac{q_0 l}{2} - Q_2\, T_2 = 0$$

$$\frac{k}{l}(T_2 - T_1) = \frac{q_0 l}{2} + Q_2\, T_2 \qquad \text{... (1.43)}$$

Arrange the equation (1.42) and 1.43) in matrix form, we get,

$$\frac{k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{q_0 l}{2} \\ \dfrac{q_0 l}{2} \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \qquad \text{... (1.44)}$$

[k] {u} = {F}

1.20 ADVANTAGES OF FINITE ELEMENT METHOD

1. One of the major advantages of FEM over approximate methods is the fact that FEM can handle irregular geometry in a convenient manner.
2. It handles general load conditions without difficulty.
3. Non-homogeneous materials can be handled easily.
4. All the various types of boundary conditions are handled.
5. Dynamic effects are included.
6. Very the size of the elements to make it possible for using small elements where necessary.
7. Higher order elements may be implemented.

8. Altering the element model with different loads, boundary conditions and other changes in the model can be done easily and cheaply.

1.21 DISADVANTAGES OF FEM

1. It requires a digital computer and fairly extensive software.
2. It required longer execution time compared with finite difference method.
3. Output result will vary considerably, when the body is modeled with fine mesh when compared to body modeled with course mesh.
4. In finite difference method, the governing differential equation of the phenomenon must be known whereas finite element method does not require to express fully.

1.22 APPLICATIONS OF FINITE ELEMENT ANALYSIS

The finite element can be used to analysis both structural and non-structural problems.

In structural problems, displacement at each nodal point is obtained. Bu using these displacement solutions, stress and strain in each element can be calculated.

Typical structural problems include:

1. Stress analysis including truss and frame analysis.
2. Stress concentration problems typically associated with holes, fillets or other changes in geometry in a body.
3. Buckling analysis: Example: Connecting rod subjected to axial compression.
4. Vibration analysis: Example: A beam subjected to different types of loading.

In non-structural problems, temperature or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow, etc., for each element can be calculated.

Non – structural problems include:

1. Heat transfer analysis:

   Example: Steady state thermal analysis on composite cylinder.

2. Fluid flow analysis.

Example: Fluid flow through pipes.

3. Distribution of electric or magnetic potential

   Example: Modeling of electromagnetic field of motor.

Recently finite element analysis is used in some biomechanical engineering problems (which may include stress analysis) typically include analysis of human spine, skull, hip joints, heart, eye, etc.

UNIT 2

ONE DIMENSIONAL PROBLEMS

2.1. INTRODUCTION

Bar and beam elements are considered as one dimensional elements. These elements are often used to model trusses and frame structures.

A bar is a member which resist only axial loads, whereas a beam can resist axial, lateral, and twisting loads. A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements.

In this unit, one dimensional elements and step-by-step procedure for the analysis of bars, trusses and beams are discussed. The total potential energy, stress-strain and strain- displacement relationships are used in developing the finite element method for a one dimensional problem. The basic one dimensional procedure is same for two and three dimensional problems.

2.2. STRESS, STRAIN, DISPLACEMENT AND LOADING

In one dimensional problems, stress (σ), strain (e), displacement (u) and loading depends only on the variable x. So, the vectors u, σ and e can be written as,

$$u = u(x)$$

$$\sigma = \sigma(x)$$

$$e = e(x)$$

The stress-strain relationship is given by.

$$\sigma = Ee$$

where,

$\sigma \rightarrow$ Stress, N/mm²

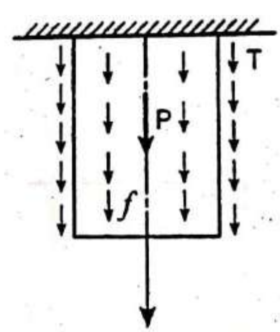


Fig. 2.1 A bar is subjected to loading

e → Strain.

E→ Young's modulus, N/mm^2

The strain-displacement relationship is given by,

$$e = \frac{du}{dx}$$

The differential volume can be written as,

dV = A dx

There are three types of loading acts on the body. They are:

(i) Body force (f).

(ii) Traction force (T).

(iii) Point load (P).

Body Force (f)

A body force is a distributed force acting on every elemental volume of the body. Unit: Force per unit volume.

Example: Self weight due to gravity.

Traction Force (T)

A traction force is a distributed force acting on the surface of the body. Unit: Force per unit area but for one dimensional problems, unit is force per unit length.

Examples: Frictional resistance, viscous drag, surface shear, etc.

Point Load (P)

Point load is a force acting at a particular point which causes displacement.

2.3. FINITE ELEMENT MODELLING

Finite element modelling consists of the following:

(i) Discretization of structure.

(ii) Numbering of nodes.

(i) Discretization

The art of subdividing a structure into a convenient number of smaller components known as discretization.

Consider a bar as shown in Fig.2.2. The first step is to model the bar as a stepped at Let us model the bar using 5 finite elements, each having a uniform cross section as shown Fig.2.3. Every element connects two nodes. Five element, six node model element is shown in Fig.2.4.

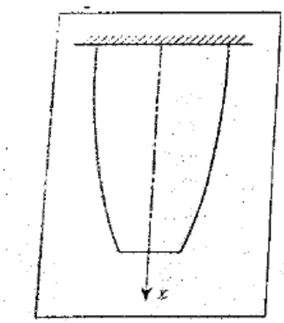


Fig. 2.2

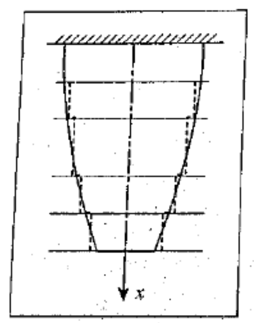


Fig. 2.3

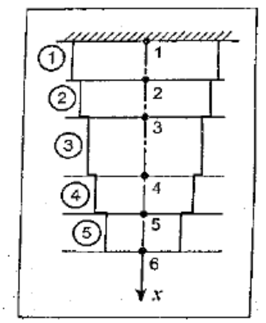


Fig. 2.4

The element numbers are circled to distinguish them from node numbers. The cross-sectional area, traction forces and body forces are constant within each element. But, these are differing in magnitude from element to element. Better results are obtained by increase number of finite elements.

(ii) Numbering of nodes

In one dimensional problem, each node is allowed to move only in $\pm x$ direction. So, each node has one degrees of freedom. (Degrees of freedom is nothing but a nodal displacement). A six node finite element model is shown in Fig.2.5. It has six degrees of freedom. Load is considered as positive if it is acting along the $+x$ direction.

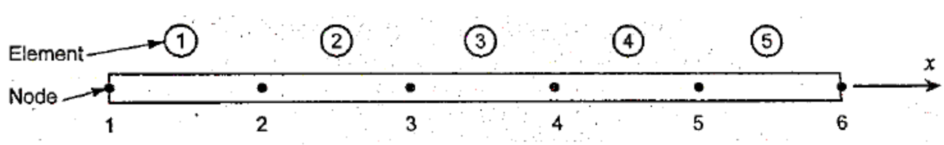


Fig. 2.5

In the element connectivity table, the heading 1 and 2 refer to local node numbers of an element and the corresponding node numbers on the structure are called global numbers. Connectivity thus establishes the local-global correspondence.

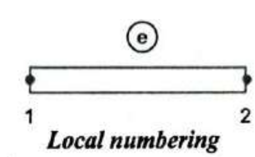


Fig. 2.6 (a)

Element (e)	Nodes 1 2
1	1 2
2	2 3
3	3 4
4	4 5
5	5 6

Local numbers

Global numbers

Fig. 2.6 (b) Connectivity bar

2.4. CO-ORDINATES

The co-ordinates are generally classified as follows:

(i) Global co-ordinates,

(ii) Local co-ordinates.

(iii) Natural co-ordinates.

2.4.1. Global Co-ordinates

The points in the entire structure are defined using co-ordinate system is known as co-ordinate system.

Example:

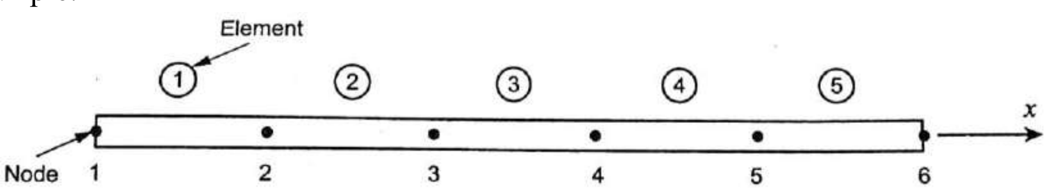


Fig. 2.7 One dimensional bar

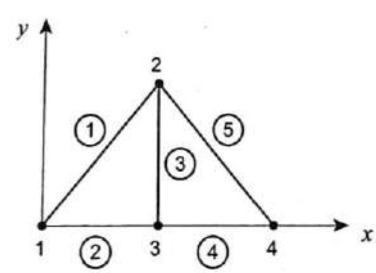


Fig. 2.8 Two dimensional triangular element

2.4.2. Local Co-ordinates

In finite element method, separate co-ordinate is used for each element. It is very used for deriving element properties. But the final equations are to be formed only by global ordinate systems.

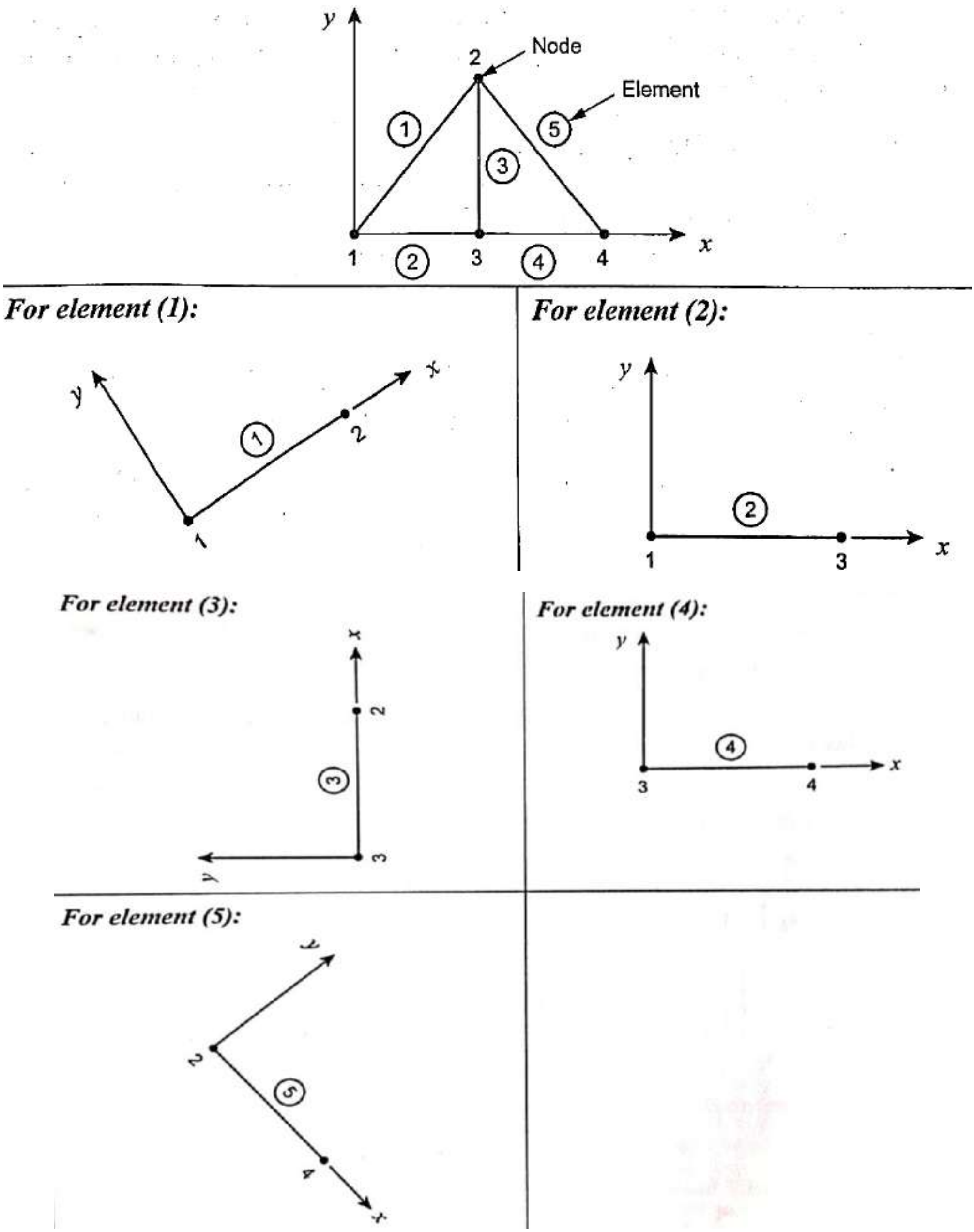


Fig. 2.9 Local co-ordinates system

2.4.3. Natural Co-ordinates

A natural co-ordinate system is used to define any point inside the element by a set of dimensionless numbers whose magnitude never exceeds unity. This system is very useful in assembling of stiffness matrices.

(1) Natural Co-ordinates in One Dimension

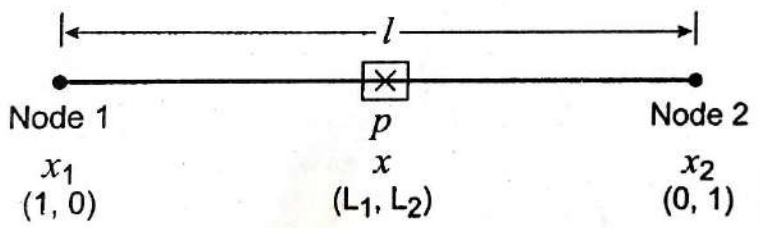


Fig. 2.10. Natural co-ordinates for a line element

Consider a two noded line element as shown in Fig.2.10. Any point p inside the line element is identified by two natural co-ordinates L1 and L2 and the Cartesian co-ordinate x. Node 1 and node 2 have the Cartesian co-ordinates x1 and x2 respectively.

We know that,

Total weightage of natural co-ordinates at any point is unity.

i.e.. $L_1 + L_2 == 1$

Any point x within the element can be expressed as a linear combination of the nodal co-ordinates of nodes 1 and 2 as,

$$L_1 x_1 + L_2 x_2 = x$$

Arrange equation (2.1) and (2.2) in matrix form,

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & 1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\left[Note: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{(a_{11}.a_{22}) - (a_{12}.a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 & - & x \\ -x_1 & + & x \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 & - & x \\ x & - & x_1 \end{Bmatrix}$$

$$= \frac{1}{l} \begin{Bmatrix} x_2 & - & x \\ x & - & x_1 \end{Bmatrix}$$

$$\{\because x_2 - x_1 \; is \; the \; length \; of \; the \; element, l\}$$

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_2 - x}{l} \\ \dfrac{x - x_1}{l} \end{Bmatrix}$$

The variation of L_1 and L_2 is shown in Fig.2.12 and Fig.2.13. L_1 is one at node 1 and it is zero at node 2 whereas L_2 is one at node 2 and it is zero at node 1.

Fig. 2.11

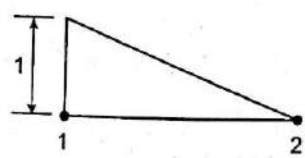


Fig. 2.12. Variation of L_1

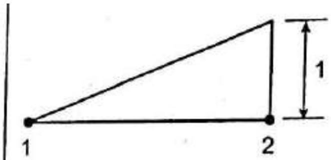


Fig. 2.13 Variation of L_2

Integration of polynomial terms in natural co-ordinates can be performed by using the simple formula,

$$\int_{x_1}^{x_2} (L_1)^{\alpha} \, (L_2)\alpha \;\; dx = \frac{\alpha! \; \beta!}{(\alpha + \beta + 1)!} \times l_x$$

Where, α! is the factorial of α.

Natural Co-ordinate, ε

In one dimensional problem, the following type of natural co-ordinate is also used. Consider a one dimensional element as shown in Fig.2.14.

In the local number scheme, the first node will be numbered 1 and the second node 2. c is the Centre of nodes and 2 and p is the point referred.

The natural coordinator ε for any point in the element is defined as,

$$\varepsilon = \frac{p\,c}{\left(\frac{x_2 - x_1}{2}\right)}$$

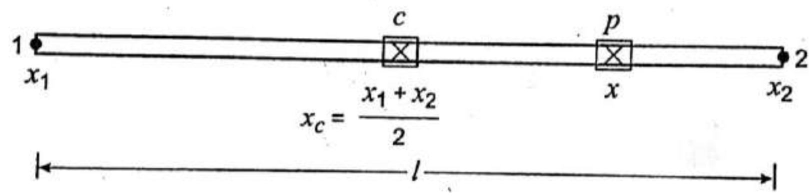


Fig. 2.14.

$$\varepsilon = \frac{p\,c}{\frac{l}{2}} \qquad [\because\ x_2 - x_1 = l]$$

$$= \frac{2}{l} p\,c = \frac{2}{l}\ (x - x_c) \qquad [\because pc = x - x_c]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_1 + x_2}{2}\right)\right] \qquad [\because x_c = \frac{x_1 + x_2}{2}]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_2 + x_1}{2}\right)\right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_2 - x_1 + 2\,x_1}{2}\right)\right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{l + 2\,x_1}{2}\right)\right]$$

$$\varepsilon = \frac{2}{l}\left[x - \left(\frac{l}{2} + x_1\right)\right]$$

$$\Rightarrow \qquad \frac{\varepsilon\, l}{2} = x - \frac{l}{2} - x_1$$

$$\Rightarrow \qquad \frac{\varepsilon\, l}{2} + \frac{l}{2} = x - x_1$$

$$\Rightarrow \quad \frac{l}{2}(\varepsilon + 1) = x - x_1 \qquad \ldots (2.4)$$

Applying boundary conditions,

At node 1, $x = x_1$

$$(2.4) \Rightarrow \quad \frac{1}{2}(1+\varepsilon) = 0$$

$$\Rightarrow \quad 1+\varepsilon = 0$$

$$\Rightarrow \quad \varepsilon = -1$$

At node 2, $x = x_2$

$$(2.4) \Rightarrow \quad \frac{l}{2}(1+\varepsilon) = x_2 - x_1$$

$$\Rightarrow \quad \frac{l}{2}(1+\varepsilon) = l$$

$$\Rightarrow \quad 1+\varepsilon = 2$$

$$\Rightarrow \quad \varepsilon = 1$$

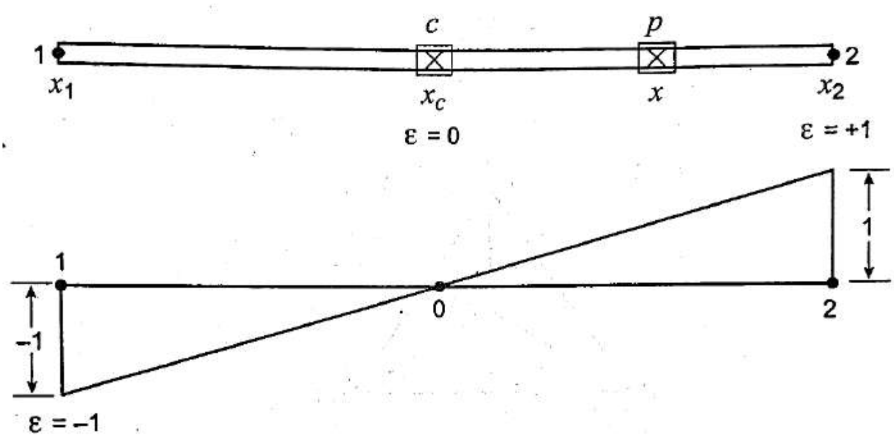


Fig. 2.15 Variation of natural co-ordinate, ε

Natural Co-ordinates in Two Dimensions

Consider a triangular element having 3 nodes as shown in Fig.2.16.

Let p is the point inside the element and it has 3 co-ordinates L_1, L_2 and L_3

From the definition of natural co-ordinates, we know that,

$$L_1 + L_2 + L_3 = 1 \qquad \text{... (2.5)}$$

$$L_1x_1 + L_2x_2 + L_3x_3 = x \qquad \text{... (2.6)}$$

$$L_1y_1 + L_2y_2 + L_3y_3 = y \qquad \text{... (2.7)}$$

Assemble the above equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad \text{... (2.8)}$$

$$Let\ D = \begin{bmatrix} + & - & + \\ 1 & 1 & 1 \\ - & + & 1 \\ x_1 & x_2 & x_3 \\ + & - & + \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$D^{-1} = \frac{C^T}{|D|} \qquad \text{... (2.9)}$$

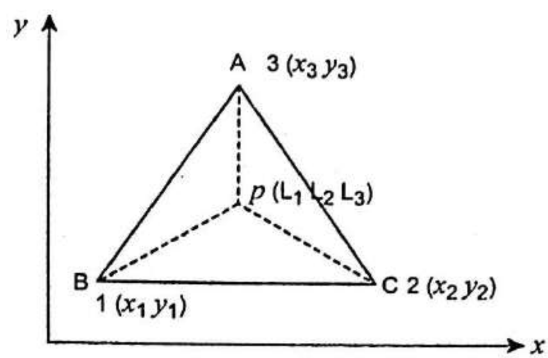


Fig. 2.16.

Coefficients of matrix D:

$$C = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_3 - x_2 & x_2 - x_1 \end{bmatrix}$$

$$\Rightarrow \qquad C^T = \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2\,y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \qquad \ldots.(2.10)$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$|D| = 1\,(x_2y_3 - x_3y_2) - 1\,(x_1y_3 - x_3y_1) + 1(x_1y_2 - x_2y_1) \qquad \ldots(2.11)$$

Substitute C^T and |D| values in equation (2.9)

$$(2.9) \Rightarrow D^{-1} = \frac{1}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)} \times \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2\,y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

Substitute D-1 values in equation (2.8),

$$\Rightarrow \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)} \times \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2\,y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad \ldots(2.12)$$

The area of the triangle ABC can be expressed as a function of the x , y co-ordinates of the nodes 1 , 2 and 3.

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}[1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)]$$

$$= \frac{1}{2}[x_2y_3 - x_3y_2 - x_1y_3 - x_1y_2 + x_3y_1 - x_2y_1]$$

$$A = \frac{1}{2}[x_2y_3 - x_3y_2 - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)]$$

$$\Rightarrow \quad (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) = 2A \quad \text{... (2.13)}$$

Substitute (2.13) value in equation (2.12),

$$\Rightarrow \quad \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad \text{... (2.12)}$$

Integration of polynomial terms in natural co-ordinates for two dimensional elements can be performed by using the formula,

$$\oint_A (L_1)^{\propto}(L_2)^{\beta}(L_3)^{\gamma}\, dA = \frac{\alpha!\ \beta!\ \gamma!}{(\alpha + \beta + \gamma + 2)!} \times 2A \quad \text{... (2.14)}$$

2.4.4. Solved problems on natural co-ordinates

Example 2.1

Calculate the value of $\oint_A L_1\, L_2\, L_3\ dA$.

Solution:

$$\oint_A L_1\, L_2\, L_3\ dA \quad \text{... (1)}$$

We know that

$$\oint_A (L_1)^{\propto}(L_2)^{\beta}(L_3)^{\gamma}\, dA = \frac{\alpha!\ \beta!\ \gamma!}{(\alpha + \beta + \gamma + 2)!} \times 2A \quad \text{... (2)}$$

Compare equation (1) and (2),

$\alpha = 1,\ \beta = 1, \gamma = 1.$

$$\oint_A L_1\, L_2\, L_3\ dA = \frac{1!\ 1!\ 1!}{(1+1+1+2)!} \times 2A$$

$$= \frac{1 \times 1 \times 1}{5!} \times 2A$$

$$= \frac{1}{5 \times 4 \times 3 \times 2 \times 1} \times 2A$$

$$\oint_A L_1\, L_2\, L_3 \;\; dA = \frac{A}{60}$$

Example 2.2

Determine the value of $\oint_A L_1(\,L_2\,)^2(L_3)^3 \;\; dA$.

Solution: we know that, $\oint_A L_1(\,L_2\,)^2(L_3)^3 \;\; dA$.

$$\oint_A^A (L_1)^{\propto}(L_2)^{\beta}(L_3)^{\gamma}\, dA = \frac{\alpha!\;\beta!\;\gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A$$

Here, $\propto = 1,\;\; \beta = 2\; and\; \gamma = 3$.

$$\oint_A \;\; L_1\, L_2^2\, L_3^3 \;\; dA = \frac{1! \times\; 1!\; \times 1!}{(1+2+3+2)!} \times 2A = \frac{1! \times\; 2!\; \times 3!}{8!} \times 2A$$

$$= \frac{1 \times 2\; \times\; 1 \times 3\; \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4\; \times 3\; \times 2 \times 1} \times 2A$$

$$\oint_A \;\; L_1\, L_2^2\, L_3^3 \;\; dA = \frac{A}{1680}$$

Example 2.3

Calculate the value of $\oint_0^1 L_1 L_2\; dx$.

Solution: we know that,

$$\oint_{x_1}^{x_2} L_1^{\propto} L_2^{\beta} dx = \frac{\alpha!\;\beta!}{(\alpha+\beta+1)!} \times l \quad \text{[From equaiton no. 2.3]}$$

Here, $\propto = 1,\;\; \beta = 1$

$$\oint_0^l L_1\, L_2 dx = \frac{1! \times 1!}{(1+1+1)!} \times l = \frac{1 \times 1}{3!} \times l = \frac{1}{3 \times 2 \times 1} \times l$$

$$\oint_A \;\; L_1\, L_2 dx \;\; = \frac{l}{6}$$

Example 2.4

Determine the value of $\int_0^1 L_1^3 dx$.

Solution: we know that,

$$\int_{x_1}^{x_2} L_1^{\alpha} L_2^{\beta}\, dx = \frac{\alpha!\ \beta!}{(\alpha + \beta + 1)!} \times l$$

Here, $\propto = 3,\ \beta = 0$

$$\oint_0^l L_1^3\, L_2^0 dx = \frac{3!\,0!}{(3+0+1)!} \times l = \frac{(3 \times 2\ \times 1)}{4!} \times l$$

$$= \frac{3 \times 2\ \times 1}{4 \times\ 3 \times 2 \times 1} \times l$$

$$\oint_0^l L_1^3\, L_2^0 dx = \frac{l}{4}$$

2.5 SHAPE FUNCTIONS

2.5.1. Introduction

If the values of the field variable are computed only at nodes, how are values obtained at other nodal points within a finite element? This is a most important point of finite element analysis.

The values of the field variable computed at the nodes are used to approximate the values at non-nodal points by interpolation of the nodal values.

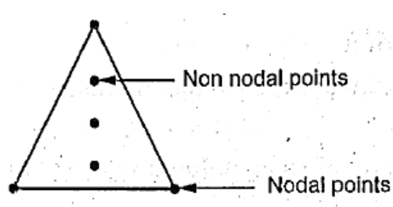


Fig.2.17

Consider the three noded triangular element as shown in Fig.2.17.

The nodes are exterior and at any point within the element the field variable is described by the following approximate relation.

$$\emptyset(x,y) = N_1(x,y)\emptyset_1 + N_2(x,y)\emptyset_2 + N_3(x,y)\emptyset_3$$

where $\emptyset_1, \emptyset_2, \emptyset_3$ are the values of the field variable at the nodes, and N_1, N_2 and N_3 are the interpolation functions. N_1, N_2 and N_3 are also called as shape functions because they are used to express the geometry or shape of the element. Shape function has unit value at one nodal point and zero value at other nodal points.

In one dimensional problem, the basic field variable is displacement.

$$u = \sum N_i u_i$$

For two noded bar element, the displacement at any point within the element is given by,

$$u = \sum N_i u_i = N_1 u_1 + N_2 u_2$$

where u_1 and u_2 are nodel displacements.

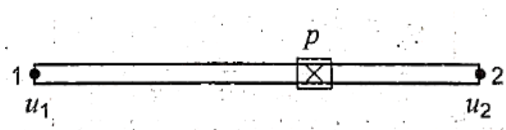


Fig. 2.18

In two dimensional stress analysis problem, the basic field variable is displacement.

So, $$u = \sum N_i u_i$$

$$v = \sum N_i v_i$$

For three noded triangular element, the displacement at any point within the element is given by,

$$u = \sum N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$u = \sum N_i u v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Where, u_1, u_2, u_3, v_1, v_2, v_3 are nodal displacements.

In general, shape functions need to satisfy the following:

1. First derivatives should be finite within an element.
2. Displacement should he continuous across the element boundary.

The characteristics of shape function are:

1. The shape function has unit value at its own nodal point and zero value at other node points.
2. The sum of shape function is equal to one.
3. The shape functions for two dimensional elements are zero along' each side that the node does not touch.
4. The shape functions are always polynomials of the same type as the original interpolation equations.

2.5.2. Polynomial Shape Functions

Polynomials are generally used as shape function due to the following reasons.

1. Differentiation and integration of polynomials are quite easy.
2. It is easy to formulate and computerize the finite element equations.
3. The accuracy of the results can be improved by increasing the order of the polynomial.

The approximation of a non-linear one dimensional function by using polynomials of different order is shown in Fig.2.19.

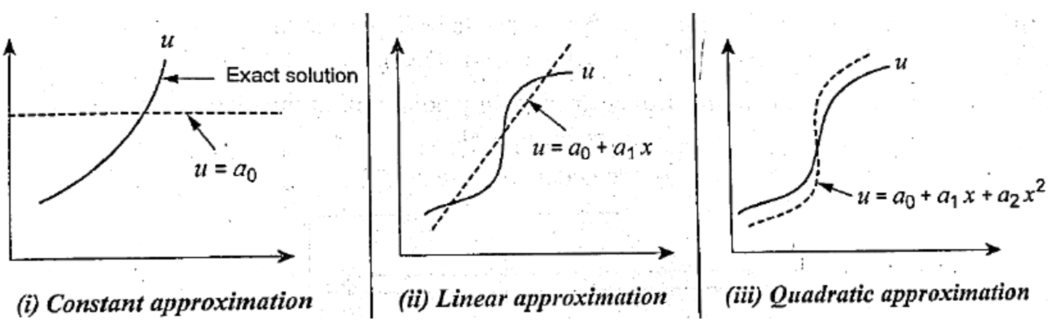


Fig. 2.19 Approximation of a function by polynomials of different order

Let us consider displacement u is a field variable.

Case (i): linear polynomial

For one dimensional problem, $u = a_0 + a_1 x$

For two dimensional problem, $u(x, y) = a_0 + a_1 x + a_2 y$

For three dimensional problem, $u(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z$

Case(ii): Quadratic polynomial

For one dimensional problem,

$$u = a_0 + a_1x + a_2x^2$$

For two dimensional problem,

$$u(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy$$

For three dimensional problem,

$$u(x, y, z) = a_1 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7\,xy + a_8\,yz + a_9\,xz$$

2.5.3. Derivation of the displacement function u and shape function N for one dimensional Linear bar element based on global co-ordinate approach

Consider a bar element with nodes 1 and 2 as shown in Fig.2.20. u_1 and u_2 are the displacements at the respective nodes. So, u_1 and u_2 are considered as degrees of freedom of this bar element.

[Note: Degrees of freedom is nothing but nodal displacements.]

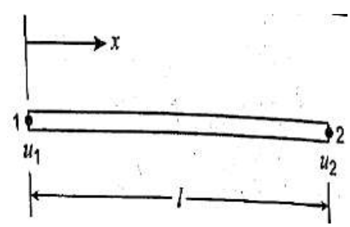


Fig 2.20 Two noded bar element

Since the element has got two degrees of freedom, it will have two generalized co- ordinates.

$$u = a_0 + a_1x \qquad \text{...(2.15)}$$

a_0 and a_1 are global or generalized co-ordinates.

Writing the equation (2.15) in matrix form,

$$u = [1\ x]\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \qquad \text{..(2.16)}$$

$At\ node\ 1, u = u_1, x = 0$

$At\ node\ 2, u = u_2, x = l$

Substitute the above value in equation (2.15),

$$u_1 = a_0 \quad \text{... (2.17)}$$

$$u_2 = a_0 + a_1 l \quad \text{... (2.18)}$$

Arranging the equation (2.17),(2.18) in matrix form,

$$\underset{u^*}{\underset{\downarrow}{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}} = \underset{C}{\underset{\downarrow}{\begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1}}} \underset{A}{\underset{\downarrow}{\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}}} \quad \text{... (2.19)}$$

Where u* → Degrees of freedom.

C → Connectivity matrix.

A → Generalized or global co-ordinates matrix.

$$\Rightarrow \qquad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l-0} \begin{bmatrix} 1 & -0 \\ -1 & l \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\left[Note \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$\Rightarrow \qquad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} 1 & 0 \\ -1 & l \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in equation (2.16),

$$\Rightarrow \qquad u = [\,1 \quad x\,] \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [\,1 \quad x\,] \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [\,1 \quad x \qquad 0 + x\,] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[\because Matrix\ multiplicatioon\ (1 \times 2) \times (2 \times 2) = (1 \times 2)]$$

$$u = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{... (2.20)}$$

$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Displacement function

$$[u = N_1 u_1 + N_2 u_2] \qquad \text{.... (2.21)}$$

Where, shape function, $N_1 = \frac{l-x}{l}$; Shape function, $\mathrm{N}_2 = \frac{x}{l}$

We may note that N_1 and N_2 obey the definition of shape function, i.e., the shape function will have a value equal to unity at node to which it belongs and zero value at other nodes

Checking: At node 1, x=0.

$$\Rightarrow \qquad N_1 = \frac{l-x}{l} = \frac{l-0}{l}$$

$$N_1 = 1$$

$$\Rightarrow \qquad N_2 = \frac{x}{l} = \frac{0}{l}$$

$$N_2 = 0$$

At node 2, x =1

$$\Rightarrow \qquad N_1 = \frac{l-x}{l} = \frac{l-0}{l}$$

$$N_1 = 0$$

$$\Rightarrow \qquad N_2 = \frac{x}{l} = \frac{0}{l}$$

$$N_2 = 1$$

2.6. THE POTENTIAL-ENERGY APPROACH

We know that

The general expression for the potential energy is given by,

$$\pi = \frac{1}{2}\int_{l} \sigma^T e \, A \, dx - \int_{l} u^T f A \, dx - \int_{l} u^T T dx - \sum_{l} u_i P_i$$

Where, σ → Stress, N/mm^2

e → Strain

A → Area, mm^2

u → Displacement, mm

f → Body force, N

T → Traciton force, N

P → Point load, N

When the continuum has been discretized into finite elements, the expression for π becomes as follows:

$$\pi = \sum_{e} \frac{1}{2}\int_{e} \sigma^T e \, A \, dx - \sum_{e}\int_{e} u^T f A \, dx - \sum_{e}\int_{e} u^T T \, dx - \sum_{l} Q_i P_i$$

The above equation can be written as,

$$\pi = \sum_{e} U_e - \sum_{e}\int_{e} u^T f A \, dx - \sum_{e}\int_{e} u^T T \, dx - \sum_{l} Q_i P_i \quad \ldots.(1)$$

where, strain energy, $U_e = \frac{1}{2}\int \sigma^T e \, A \, dx$

Stiffness matrix for a bar element:

We know that,

Strain energy, $U_e = \frac{1}{2}\int \sigma^T e \, A \, dx$ ...(2)

we know that,

Strain, $e = B\,u$

Strain, $\sigma = E\,e$ $\left[Young's\ modulus, E = \dfrac{Stress, \sigma}{Strain, e}\right]$

$\Rightarrow Strain, \sigma = E \times Bu$

Subsitute σ and e values in equation (2),

$$\Rightarrow U_e = \frac{1}{2}\int (E\,B\,u)^T\,(B\,u)\,A\,dx = \frac{1}{2}\int E\,B^T\,u^T B\,u\,A\,dx$$

$$= \frac{1}{2}u^T \int EB^T\,B\,u\,A\,dx$$

$$= \frac{1}{2}u^T \left[A\,E\,B^T\ \ B\,u \int_0^1 dx\right]$$

$$= \frac{1}{2}u^T\,[A\,E\,B^T\ \ B\,u\,[x]_0^l]$$

$$U_e = \frac{1}{2}u^T\,[A\,E\,B^T\ \ B\,u\,(l)] \qquad \ldots(3)$$

From equation (2.31), we know that,

Strain displacement matrix,

$$[B] = \begin{bmatrix}\dfrac{-1}{l} & \dfrac{1}{l}\end{bmatrix}$$

$$\Rightarrow \qquad [B]^T = \begin{Bmatrix}\dfrac{-1}{l} \\ \dfrac{1}{l}\end{Bmatrix}$$

Substitute B, B^T values in equation (2.94)

$$\Rightarrow \qquad U_e = \frac{1}{2}u^T.AE\begin{Bmatrix}\dfrac{-1}{l} \\ \dfrac{1}{l}\end{Bmatrix}\begin{bmatrix}\dfrac{-1}{l} & \dfrac{1}{l}\end{bmatrix}u\,l$$

$$= \frac{1}{2} u^T . AE \times \frac{1}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times \frac{1}{l} [-1 \ 1] u \ l$$

$$= \frac{1}{2} u^T \frac{AE}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} [-1 \ 1] u$$

$$U_e = \frac{1}{2} u^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

$$[\because (2 \times 1) \times (1 \times 2) = 2 \times 2]$$

The above equation is in the form of

$$U_e = \frac{1}{2} u^T [K] u$$

Where, Stiffness matrix,

$$[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{... (4)}$$

2.7. STIFFNESSS MATRIX [K]

In order to get an expression for the stiffness matrix in finite element method, let us review the strain energy expression in structural mechanics.

Consider $\omega_1, \omega_2 \ldots \omega_n$ are nodal displacement parameters or otherwise known as degrees of freedom, W1,W2,…Wn are the corresponding nodal loads acting at degrees of freedom. {ω} and {W} are column matrix.

$$\{W\} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_n \end{Bmatrix}$$

$$\{\omega\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \end{Bmatrix}$$

We know that, $\{W\} = [K] \{\omega *\}$ …… (2.22)

Where, W = Nodal loads.

K = stiffness matrix.

ω = degrees of freedom

From equation (2.22), we know that, nodal loads and the corresponding degrees of freedom of freedom are linked through stiffness matrix.

We know that.

Work done, P= strain energy

$$\Rightarrow \qquad P = \frac{1}{2}\, W_1\omega_1 + \frac{1}{2} W_2\omega_2 + \frac{1}{2} W_3\omega_3 + \cdots \ldots \ldots + \frac{1}{2} W_n\omega_n$$

We can write this equation in matrix form,

$$\text{i.e.,} \qquad P = \frac{1}{2}[W_1\ W_2\ W_3 \ldots \ldots .. W_n] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \end{Bmatrix}$$

$$P = \frac{1}{2}\{W\}^T\,\{\omega^*\} \qquad \ldots (2.23)$$

[Note: [ ] → Row matrix; { } → Column matrix.

$$P = \frac{1}{2}\{[\mathrm{K}]\{\omega\}\}^T\,\{\omega^*\}$$

$$P = \frac{1}{2}[K]^T\,\{\omega\}^T\,\{\omega^*\}$$

$$P = \frac{1}{2}\,\{\omega^*\}^T[K]\,\,\{\omega^*\} \qquad \ldots (2.24)$$

[K is a symmetric matrix. So, $[K]^T = [K]$]

Equation (2.24) is a strain energy equation for a structure.

Our aim is to find the expression for stiffness matrix [K]. Let us consider one dimensional element. u_1, u_2, u_3,…..,u_n are the degrees of freedom of that element.

We know that,

$$\text{Strain,} \qquad \{e\} = [B]\,\{u^*\} \qquad \ldots (2.25)$$

$$\Rightarrow \qquad \{e\}^T = [B]^T\{u^*\}^T \qquad \ldots (2.26)$$

Where, { e} is a strain matrix [Column matrix].

[B] is a strain-displacement matrix [Row matrix].

{ u*} is a degree of freedom [column matrix].

We know that,

$$Stress \qquad \{\sigma\} = [E]\{ e\}$$

$$\{\sigma\} = [D]\{ e\} \qquad \text{... (2.27)}$$

$Where, [E] = [D] = Young's\ modulus.$

Strain energy expression is given by,

$$U = \int_{v} \frac{1}{2} \{ e\}^{T} \{\sigma\} \ \ dv \qquad \text{... (2.28)}$$

Substitute $\{e\}^T$ and $\{\sigma\}$ values,

$$\Rightarrow \qquad U = \int_{v} \frac{1}{2} [B]^{T} \{u^{*}\}^{T} [D]\{e\}\, dv$$

$$\frac{1}{2}\{u^{*}\}^{T} \int_{v} \frac{1}{2} [B]^{T} [D]\{e\}\, dv$$

Substitute {e } value,

$$\Rightarrow \qquad U = \frac{1}{2} \{ u^{*}\}^{T} \int_{v} \ [B]^{T} \ [D][B] \ \{u^{*}\}\, dv$$

$$U = \frac{1}{2} \{ u^{*}\}^{T} \left[\int_{v} \ [B]^{T} \ [D][B] \ \{u^{*}\}\, dv\right] \{u^{*}\} \qquad \text{... (2.29\}}$$

From equation (2.24), we know that,

$$P = \frac{1}{2}\{\omega^{*}\}^{T} [K]\{ \omega^{*}\} \qquad \text{... (2.24)}$$

Comparing equation (2.29) and (2.24),

$$\Rightarrow \qquad \{\omega^{*}\}^{T} = \{u^{*}\}^{T}$$

$$\{\omega^{*}\} \quad = \{u^{*}\}$$

$$[K] = \int_v [B]^T [D][B] dv$$

So, stiffness matrix, $[K] = \int_v [B]^T [D][B] dv$ ... (2.30)

Where, [B] → strain displacement relationship matrix.

[D] → Elasticity matrix or Stress-strain displacement relationship matrix.

In one dimensional problem,

Strian, $e = \frac{du}{dx}$

Where, u → Displacement function.

[D] = [E] = E= Young's modulus.

In Beam problem, Strain,

$$e = Curvature = \frac{d^2u}{dx^2}$$

[D] = [E I] = Flexural rigidity.

2.7.1. Properties of Stiffness Matrix

1. It is a symmetric matrix.
2. The sum of elements in any column must be equal to zero.
3. It is an unstable element. So, the determinant is equal to zero.
4. The dimension of the global stiffness matrix [K] is N x N, where N is the number of nodes. This follows from the fact that each node has only one degree of freedom.
5. The diagonal coefficients are always positive and relatively large when compared to the off-diagonal values in the same row.

2.7.2. Derivation of Stiffness Matrix for One Dimensional Linear Bar Element

Consider a one-dimensional bar element with nodes 1 and 2 as shown in Fig.2.21. Let u_1 and u_2 be the nodal displacement parameters or otherwise known as degrees of freedom.

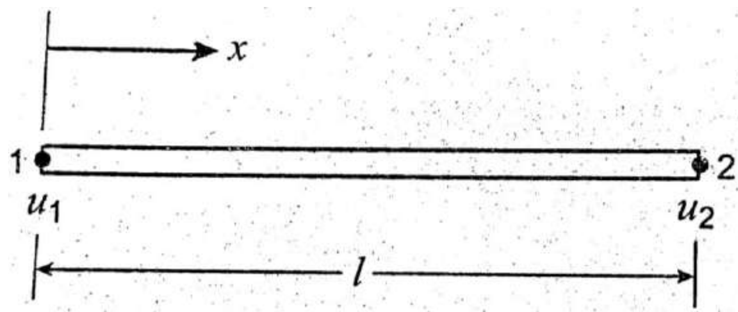


Fig. 2.21. A bar element with two nodes

We know that,

Stiffness matrix $\quad [K] = \int [B]^T[D][B]dv \quad$ [From equation no. (2.30)]

In one dimensinal bar element,

Displacement function, $u = N_1u_1 + N_2u_2$ [From equation no. (2.21)]

where, $$N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

We know that,

$$\text{Strain} - \text{Displacement matrix}, [B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \quad \text{... (2.31)}$$

$$\Rightarrow \quad [B]^T = \begin{bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{bmatrix} \quad \text{... (2.32)}$$

In one dimensional problems, [D] = [E] = E =Young's modulus …(2.33)

Substitute [B], $[B]^T$ and [D] values in stiffness matrix equation. [Limit is 0 to l].

$$\Rightarrow \qquad [K] = \int_0^l \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times E \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv = \int_0^1 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E\, dv$$

$$[\because \text{ Matrix multiplication } (2 \times 1) \times (1 \times 2) = (2 \times 2)]$$

$$= \int_0^1 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E\, A\, dx \qquad [dv = A\, dx]$$

$$= A\, E \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^1 dx = A\, E \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= A\, E \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l - 0) = A\, E\, l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$= \frac{A\, E\, l}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\, K\,] = \frac{A\, E}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \ldots (2.34)$$

The properties of a stiffness matrix are satisfied.

1. It is symmetric.
2. The sum of elements in any column is equal to zero.

2.7.3. Derivation of Finite Element Equation for one Dimensional Linear Bar Element

We know that, General force equation is,

$$\{F\} = [K]\, \{u\} \qquad \ldots (2.35)$$

where, {F} is a element force vector [Column matrix].

[K] is a stiffness matrix [Row matrix].

{u} is a nodal displacement [Column matrix].

For one dimensional bar element, stiffness matrix [K] is given by,

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{[From equation no. (2.34)]}$$

For one noded bar element,

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute [K] {F} and {u} values in equation (2.35),

$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{... (2.36)}$$

This is a Finite Element Equation for One Dimensional two noded elements.

2.7.4. Assembling The Stiffness Equations or Global Equations

Consider a bar as shown in Fig.2.22(a). This bar can be equally divided into 4 elements as shown in Fig.2.22(b).

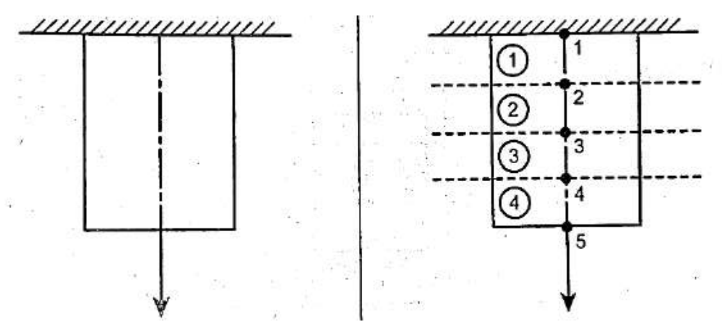


Fig. 2.22 (a) **Fig. 2.22 (b)**

Now the bar has 4 elements with 5 nodes

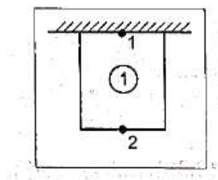
[Note: A number with circle denotes element and without circle denotes nodes].

We know that,

Finite element equation for two noded bar element is,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

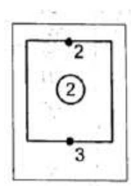
For element (1) (Nodes 1, 2):



Finite element equation is,

$$\begin{Bmatrix} F_1 \\ \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} a_{11} & a_{12} \\ 1 & -1 \\ a_{21} & a_{22} \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ \\ u_2 \end{Bmatrix} \quad \dots (2.37)$$

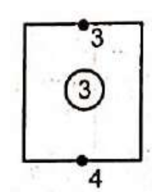
For element (2) (Nodes 2, 3):



Finite element equation is,

$$\begin{Bmatrix} F_2 \\ \\ F_3 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} a_{22} & a_{23} \\ 1 & -1 \\ a_{32} & a_{33} \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ \\ u_3 \end{Bmatrix} \quad \dots (2.38)$$

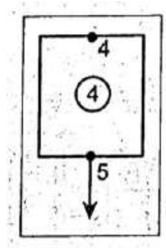
For element (3) (Nodes 3, 4):



Finite element equation is,

$$\begin{Bmatrix} F_3 \\ \\ F_4 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} a_{33} & a_{34} \\ 1 & -1 \\ a_{43} & a_{44} \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 \\ \\ u_4 \end{Bmatrix} \quad \dots (2.39)$$

For element (4) (Nodes 4, 5):



Finite element equation is,

$$\begin{Bmatrix} F_4 \\ F_5 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} a_{44} & a_{45} \\ 1 & -1 \\ a_{54} & a_{55} \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} \quad \text{... (2.40)}$$

Assembling the equations (2.37), (2.38), (2.39) and (2.40),

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 1 & -1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & q_{25} \\ -1 & 1+1 & -1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{25} \\ 0 & -1 & 1+1 & -1 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & -1 & 1+1 & -1 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

[Note : The bar has 5 nodes and each node has one degree of freedom. So, the global stiffness matrix size is 5 × 5].

$$[K]_{global} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

2.8 THE LOAD OR FORCE VECTOR {F}

Consider a vertically hanging bar of length *l*, uniform cross-section A, density ρ and young's modulus E. This bar is subjected to self-weight X_b.

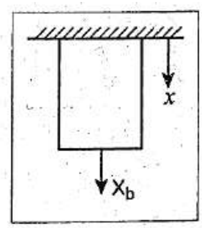


Fig. 2.23. Vertically hanging bar will self-weight

The element nodal force vector is given by,

$$\{F\}_e = \int [N]^T X_b \qquad \text{... (2.41)}$$

We know that, self weight due to loading force,

$$X_b = \rho\, A\, dx \qquad \text{... (2.42)}$$

For one dimensional bar element, the displacement function is given by,

$$u = N_1 u_1 + N_2 u_2 \qquad \text{[From equaiton no. (2.21)]}$$

where,

$$N_1 = \frac{l - x}{l}$$

$$N_2 = \frac{x}{l}$$

$$\Rightarrow \qquad [N] = \left[\frac{l-x}{l} \quad \frac{x}{l}\right]$$

$$\Rightarrow \qquad [N]^T = \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \qquad \text{... (2.43)}$$

Substitute X_b and $[N]^T$ values in equation (2.41),

$$\Rightarrow \qquad \{F\}_e = \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \rho\, A\, dx = \rho\, A \int_0^1 \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= \rho\, A \int_0^l \begin{Bmatrix} 1 - \dfrac{x}{l} \\ \dfrac{x}{l} \end{Bmatrix} dx = \rho\, A \int_0^1 \begin{Bmatrix} dx - \dfrac{x\, dx}{l} \\ \dfrac{x\, dx}{l} \end{Bmatrix}$$

$$= \rho\, A \begin{Bmatrix} x - \dfrac{x^2}{2\, l} \\ \dfrac{x^2}{2\, l} \end{Bmatrix}_0^1 = \rho\, A \int_0^1 \begin{Bmatrix} l - \dfrac{l^2}{2\, l} \\ \dfrac{l^2}{2\, l} \end{Bmatrix} = \rho\, A \begin{Bmatrix} l - \dfrac{l}{2} \\ \dfrac{l}{2} \end{Bmatrix}$$

$$= \rho\, A \begin{Bmatrix} \dfrac{l}{2} \\ \dfrac{l}{2} \end{Bmatrix}$$

Force vector, $$[F]_e = \frac{\rho\, A\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad ..(2.44)$$

2.9. SOLVED PROBLEMS ON LINEAR BAR ELEMENTS

Example 2.5

A two noded truss element is shown in fig(i). The nodal displacement are u_1=5mm and u_2=8mm. calculate the displacement at x=l/4, l/3 and l/2.

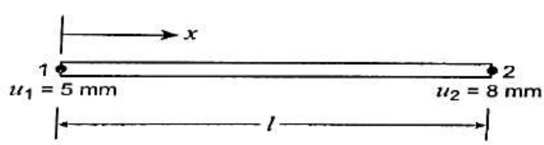


Fig. (i)

Given : Displacement , $u_1 = 5$ mm

$u_2 = 8$ mm

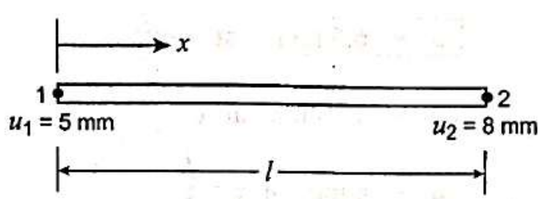


Fig. (ii)

To find: Displacement u at

$$x = \frac{l}{4}, \frac{l}{3} \; and \; \frac{l}{2}.$$

Solution: Displacement function for two noded truss element is given by,

$$u = N_1 u_1 + N_2 u_2 \qquad \text{[From equation (2.21)]}$$

where, $$N_1 = \frac{l - x}{l}$$

$$N_2 = \frac{x}{l}$$

$\Rightarrow$ $$u = \left[\frac{l - x}{l}\right] u_1 + \left[\frac{x}{l}\right] u_2 \qquad \ldots (1)$$

Substitute $x = \frac{1}{4}, u_1 = 5 \; and \; u_2 = 8 \; in$ equaiton (1),

$\Rightarrow$ $$u = \left[\frac{l - \frac{1}{4}}{l}\right] \times 5 + \left[\frac{\frac{l}{4}}{l}\right] \times 8$$

$$= \left[1 - \frac{1}{4}\right] \times 5 + \left[\frac{1}{4}\right] \times 8$$

$$u = 5.75 \; mm \; at \; x = \frac{l}{4}$$

Substitute $x = \frac{1}{3}, u_1 = 5 \; and \; u_2 = 8 \; mm \; in \; equaiton \; (1),$

(1)$\Rightarrow$ $$u = \left[\frac{l - \frac{l}{3}}{l}\right] \times 5 + \left[\frac{\frac{l}{3}}{l}\right] \times 8$$

$$= \left[1 - \frac{1}{3}\right] \times 5 + \left[\frac{1}{3}\right] \times 8$$

$$= \left[1 - \frac{1}{4}\right] \times 5 + \left[\frac{1}{4}\right] \times 8$$

$$u = 6 \; mm \; at \; x = \frac{l}{3}$$

Substitute $x = \frac{1}{2}, u_1 = 5\ and\ u_2 = 8\ mm\ in\ equaiton\ (1),$

$$(1) \Rightarrow \qquad u = \left[\frac{l - \frac{l}{2}}{l}\right] \times 5 + \left[\frac{\frac{l}{2}}{l}\right] \times 8$$

$$= \left[1 - \frac{1}{2}\right] \times 5 + \left[\frac{1}{2}\right] \times 8$$

$$u = 6.5\ mm\ at\ x = \frac{l}{2}$$

Result:

$$u = 5.75\ mm\ at\ x = \frac{l}{4}$$

$$u = 6\ mm\ at\ x = \frac{l}{3}$$

$$u = 6.5\ mm\ at\ x = \frac{l}{2}$$

Example 2.6

A one dimensional bar is shown in fig.(i). calculate the following:

(i) Shape function N_1 and N_2 at point P

(ii) If U_1=3mm and U_2=-5mm, calculate the displacement u at point P.

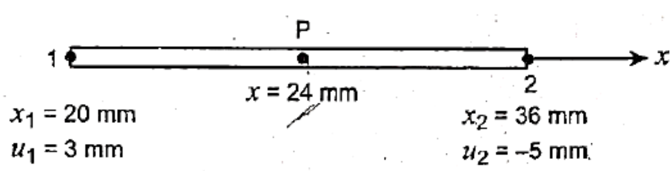


Fig. (i)

Given: $x_1 = 20\ mm,\ x_2 = 36\ mm, u_1 = 3\ mm, u_2 = -5\ mm, x = 24\ mm$

To find: 1. Shape function N_1 and N_2 at point P.

2. Displacement u at point P.

Solution:

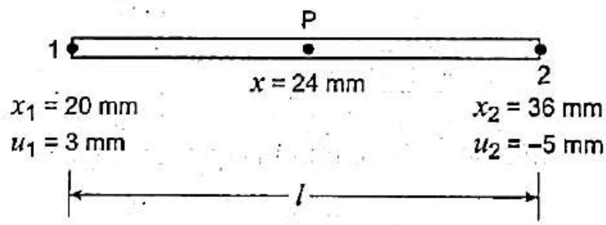


Fig. (ii)

We know that,

Actual length of the bar, $l = x_2 - x_1 = 36 - 20$

$$l = 16\ mm$$

The distance between point 1 and point P is

$$x = 24 - 20$$

$\Rightarrow$ $$x = 4 \text{ mm}$$

Displacement function for two noded bar element is given by

$$u = N_1 u_1 + N_2 u_2 \qquad \text{... (1)}$$

[From equation no.(2.21)]

where, $$N_1 = \frac{l - x}{l}$$

$$N_2 = \frac{x}{l}$$

$\Rightarrow$ $$N_1 = \frac{16 - 4}{16}$$

$$N_1 = 0.75\ mm$$

$\Rightarrow$ $$N_2 = \frac{4}{16}$$

$$N_2 = 0.25\ mm$$

Substitute $N_1, N_2, u_1\ and\ u_2 values\ in\ equaiton\ no.(1),$

(1)$\Rightarrow$ $u = N_1 u_1 + N_1 u_2 = (0.75)(3) + 0.25(-5)$

$$u = 1\ mm$$

Result: 1. Shape function, $N_1 = 0.75$ mm and $N_2 = 0.25$ mm

2. Displacement u at point P is 1mm

Example 2.7

Consider a bar as shown in fig. (i). cross-sectional area of the bar is 750mm^2 and young's modulus is $2*10^5$ N/mm^2. If u_1=0.5mm and u_2=0.625mm, calculate the following: (i) Displacement at point, P (ii) Strain, e (iii) Stress,σ (iv) Elements stiffness matrix,[K] (v)strain energy, U

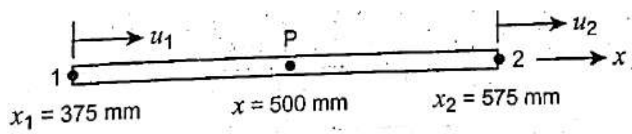


Fig. (i)

Given: Area A = 750 mm^2

Young's Modulus, E = 2×10^5 N/mm^2

Displacement, u_1 = 0.5 mm

$$u_2 = 0.625\ mm$$

Distance, $x_1 = 375\ mm$

$$x_2 = 575\ mm$$

$$x = 500\ mm$$

Solution: We know that, strain,

$$e = [B]\ \{u^*\} \qquad \text{[From equaiton no. (2.25)]}$$

Where, [B] is a strain-displacement matrix.

$\{u^*\}$ is a degree of freedom

$$\Rightarrow \qquad [B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \qquad \text{[From equaiton no. (2.31)]}$$

$$= \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix}$$

Strain, $e = [B]\{u^*\} = \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$$= \left[\frac{-1}{200}\frac{1}{200}\right]\begin{Bmatrix} 0.5 \\ 0.625 \end{Bmatrix}$$

$$= \left[\frac{-1}{200} \times 0.5 + \frac{1}{200} \times 0.625\right]$$

Strain, $e = 6.25 \times 10^{-4}$

We know that,

Strain, $\sigma = E\, e = 2 \times 10^5 \times 6.25 \times 10^{-4}$

$\sigma = 125\ N/mm^2$

For one dimensional bar element, stiffness matrix is given by,

$$[K\,] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{750 \times 2 \times 10^5}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K\,] = 7.5 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

We know that, Strain, $U = \frac{1}{2}\{u^*\}^T[\,K]\{\,u^*\}$

…….. [From equation no. (2.24)]

$$= \frac{1}{2}[\,u_1\,u_2] \times 7.5 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2}[\,0.5\ 0.625] \times 7.5 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.625 \end{Bmatrix}$$

$$= \frac{1}{2}[\,0.5\ 0.625] \times 7.5 \times 10^5 \begin{Bmatrix} 0.5 & -0.625 \\ -0.5 & +0.625 \end{Bmatrix}$$

$[\because (2 \times 2\,) \times (2 \times 1) = 2 \times 1]$

$$= \frac{1}{2} \times 7.5 \times 10^5\ [0.5\ \ 0.625]\begin{Bmatrix} -0.125 \\ 0.125 \end{Bmatrix}$$

$$= \frac{1}{2} \times 7.5 \times 10^5\ [0.5 \times (-0.125) + 0.625 \times 0.125\,]$$

Strain energy, U =5859.37 N-mm

Result:

(i) u = 0.5781 mm

(ii) $e = 6.25 \times 10^{-4}$

(iii) $\sigma = 125\ N/mm^2$

(iv) $[K] = 7.5 \times 10^5$

(v) U = 5859.37 N-mm

Example 2.8

A steel bar of length 800 mm is subjected to all axial load of 3 kN as shown in Fig. (i). Find the elongation of the bar, neglecting self weight.

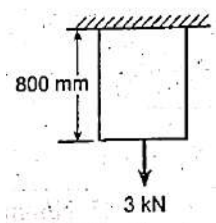


Fig. (i)

Take E = 2 × 10⁵ N/mm², A = 300 mm²

Given: Length, l = 800 mm

Load, F = 3 kN = 3×10^3 N

Young's modulus, E = $2 \times 10^5\ N/mm^2$

Area, A = $300\ mm^2$

To find: Elongation, u

Solution: we can divide the bar into two elements as shown in Fig.(ii).

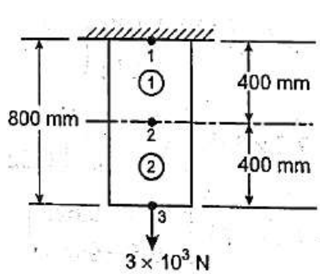
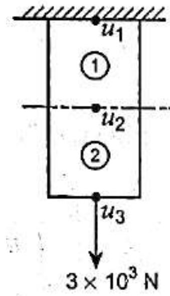


Fig. (ii)

Now the bar has 2 elements with 3 nodes.

[Note: Number with circle denotes Element & Number without circle denotes Node]

We can find the displacement at node 1, node 2 and node 3 .



Displacement at node 1 is u_1, node 2 is u_2 and node 3 is u_3.

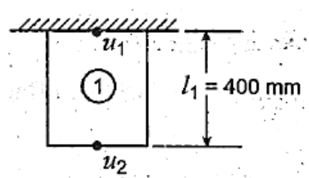
For one dimensional two noded bar element, the finite element equation is,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{[From equaiton no. (2.36)]}$$

For element 1: (Nodes 1, 2):

Finite element equation is,

$$\frac{A_1 E}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



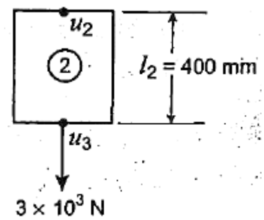
$$\frac{300 \times 2 \times 10^5}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$150 \times 10^3 \begin{bmatrix} a_{11} & a_{12} \\ 1 & -1 \\ a_{21} & a_{22} \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ \\ \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \\ \\ F_2 \end{Bmatrix} \qquad \ldots\ldots (1)$$

For element 2: (Nodes 2, 3):

Finite element equation is,

$$\frac{A_2E}{l_2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix}$$



$$\Rightarrow \quad \frac{300\times 2\ \times 10^5}{400}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix}$$

$$\Rightarrow \quad 150\times 10^3\begin{bmatrix}\overset{a_{22}}{1} & \overset{a_{23}}{-1}\\\overset{a_{32}}{-1} & \overset{a_{33}}{1}\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix} \qquad \ldots..(2)$$

Assemble the finite elements, i.e., assemble the finite element equation (1) and (2).

$$\Rightarrow 150\times 10^3\begin{bmatrix}\overset{a_{11}}{1} & \overset{a_{12}}{-1} & \overset{a_{13}}{0}\\\overset{a_{21}}{-1} & \overset{a_{22}}{1+1} & \overset{a_{23}}{-1}\\\overset{a_{31}}{0} & \overset{a_{32}}{-1} & \overset{a_{33}}{1}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix}$$

$$\Rightarrow 150\times 10^3\ \underset{\substack{\downarrow\\ [K]\ global}}{\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix} \qquad \ldots(3)$$

[Note : The rod has 3 nodes. Each node has single degree of freedom. So, the global stiffness matrix [K] size is 3 × 3.

It may be noted the stiffness matrix properties are satisfied.

$$[K\,]=\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}$$

1. It is symmetric.
2. The sum of elements in any column is equal to zero.

Applying Boundary Conditions

(i) Displacement at node 1 is zero. i.e., $u_1 = 0$.

(ii) 3×10^3 N load is acting at node 3, i.e., $F_3 = 3 \times 10^3$ N. Self-weight is neglected, so, $F_1 = F_2 = 0$.

Substitute u_1, F_1, F_2 and F_3 values in equation (3),

$$\Rightarrow \quad 150 \times 10^3 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3 \times 10^3 \end{Bmatrix}$$

Here, $u_1 = 0$. So, neglect first row and first column of [K] matrix, Hence, the final reduced equation is,

$$\Rightarrow \quad 150 \times 10^3 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \times 10^3 \end{Bmatrix}$$

$$150 \times 10^3 \, (2u_2 - u_3) = 0 \qquad \ldots (4)$$

$$150 \times 10^3 \, (-u_2 + u_3) = 3 \times 10^3 \qquad \ldots (5)$$

Solving,

$$150 \times 10^3 (u_2) = 3 \times 10^3$$

$$u_2 = 0.02 \, mm$$

Substitute u_2 value in equation (4),

$$\Rightarrow \quad 150 \times 10^3 (2 \times 0.02 - u_3) = 0$$

$$\Rightarrow \quad 2 \times 0.02 - u_3 = 0$$

$$\Rightarrow \quad 2 \times 0.02 = u_3$$

$$\Rightarrow \quad u_3 = 0.04 \, mm$$

Verification: We know that, Total elongation,

$$\delta L = \frac{pL}{AE}$$

$$= \frac{3 \times 10^3 \times 800}{300 \times 2 \times 10^5}$$

$$\delta L = 0.04 \, mm$$

Result:

1. Elongation or displacement at node 1, $u_1 = 0$
2. At node 2, $u_2 = 0.02$ mm
3. At node 3, $u_3 = 0.04$ mm

Example 2.9

Solve the problem explained in Example 2.8, by dividing the bar into 4 equal parts.

Solution:

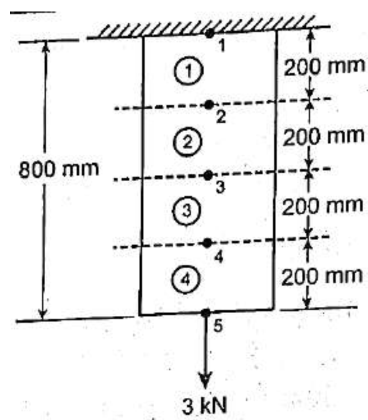


Fig. (i)

The bar has 4 elements with 5 nodes as shown in Fig.(i).

We can find the displacement at node 1, node 2, node 3, node 4 and node 5. i.e., u_1, u_2, u_3, u_4 and u_5.

Finite element equation for one dimensional two noded bar element is given by,

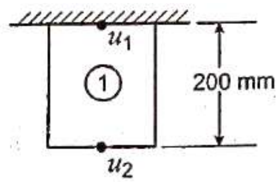
$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1 (Nodes 1, 2): Finite element equation is,

$$\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad [\because A_1 = A_2 = A_3 = A_4 = 300\ mm^2]$$

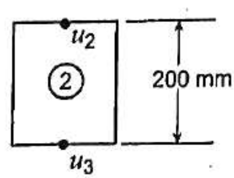
$$\frac{300 \times 2\ \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$300 \times 2 \times 10^5 \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{-1} \\ \overset{a_{21}}{-1} & \overset{a_{22}}{1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1)}$$



For element 2 (Nodes 2, 3): Finite element equation is,

$$\frac{A_2 E}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

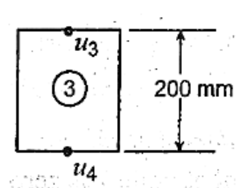


$$\frac{300 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$300 \times 10^5 \begin{bmatrix} \overset{a_{22}}{1} & \overset{a_{23}}{-1} \\ \overset{a_{32}}{-1} & \overset{a_{33}}{1} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \text{... (2)}$$

For element 3 (Nodes 3, 4):

$$\Rightarrow \qquad \frac{A_3 E}{l_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$



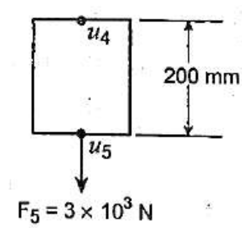
$$\Rightarrow \qquad \frac{300 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$300 \times 10^3 \begin{bmatrix} a_{33} & a_{34} \\ 1 & -1 \\ a_{43} & a_{44} \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} \qquad \ldots (3)$$

For element 4: (Nodes 4, 5):

$$\frac{A_4 E}{l_4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_5 \end{Bmatrix}$$

$$\frac{300 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_5 \end{Bmatrix}$$



$$300 \times 10^3 \begin{bmatrix} a_{44} & a_{45} \\ 1 & -1 \\ a_{54} & a_{55} \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_5 \end{Bmatrix} \qquad \ldots (4)$$

Assemble the finite elements, i.e., assemble the finite element equation (1), (2), (3) and (4)

$$300 \times 10^3 \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 1 & -1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ -1 & 1+1 & -1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & -1 & 1+1 & -1 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & -1 & 1+1 & -1 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} \qquad \ldots (5)$$

[Note: the rod has 5 nodes. Each node has single degree of freedom. So, the global stiffness matrix [K] size is 5 × 5.

It may be noted that the stiffness matrix properties are satisfied.

1. [K] matrix is symmetric.
2. The sum of the elements in any column is equal to zero.

Applying Boundary Conditions:

Displacement at node 1 is zero. i.e., $u_1 = 0$ self –weight is neglected.

So, $F_1 = F_2 = F_3 = F_4 = 0$

$3 \times 10^3 N$ *l*oad is acting at node 5 .

So, $F_5 = 3 \times 10^3 N$

Substitute u_1, F_1, F_2, F_3, F_4 *and* F_5 *values in equation* (5).

$$\Rightarrow \qquad 150 \times 10^3 \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_{[K]} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \times 10^3 \end{Bmatrix} \qquad \ldots (6)$$

In equation (6), $u_1 = 0$. So, neglect first row and first column of [K] matrix.

$$\Rightarrow \qquad 300 \times 10^3 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 3000 \end{Bmatrix} \qquad \ldots (7)$$

By using Gaussian Elimination method, we can find the solution of equation (7).

Let ,
$$\left(\begin{array}{cccc:c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 3000 \end{array}\right)$$

$$\left(\begin{array}{cccc:c} 1 & -1/2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 3000 \end{array}\right)_{R_1 \to \frac{R_1}{2}}$$

$$\left(\begin{array}{cccc:c} 1 & -1/2 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 3000 \end{array}\right)_{R_2 \to R_2 + R_1}$$

$$\left(\begin{array}{cccc:c} 1 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 3000 \end{array}\right)_{R_2 \to R_2 \times \frac{2}{3}}$$

$$\left(\begin{array}{cccc:c} 1 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1 & 1 & 3000 \end{array}\right)_{R_3 \to R_3 + R_2}$$

$$\left(\begin{array}{cccc:c} 1 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & -1 & 1 & 3000 \end{array}\right)_{R_3 \to R_3 \times \frac{3}{4}}$$

$$\left(\begin{array}{cccc:c} 1 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & -1 & 1/4 & 3000 \end{array}\right)_{R_4 \to R_4 + R_3}$$

Equation (7) becomes,

$$300 \times 10^3 \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 3000 \end{Bmatrix}$$

$$\Rightarrow \quad 300 \times 10^3 \left[0\,u_2 + 0\,u_3 + 0\,u_4 + \frac{1}{4} u_5 \right] = 3000$$

$$\Rightarrow \quad 300 \times 10^3 \left(\frac{1}{4} u_5 \right) = 3000$$

$$\Rightarrow \quad u_5 = 0.04\, mm$$

$$\Rightarrow \quad 300 \times 10^3 \left[0\,u_2 + 0\,u_3 + u_4 - \frac{3}{4} u_5 \right] = 0$$

$$\Rightarrow \quad u_4 - \frac{3}{4} u_5 = 00$$

$$\Rightarrow \quad u_4 = \frac{3}{4}(0.04)$$

$$\Rightarrow \quad u_4 = 0.03\, mm$$

$\Rightarrow \quad 300 \times 10^3 \left[0\, u_2 + 0\, u_3 - \frac{2}{3}\, u_4 + 0\, u_5 \right] = 0$

$\Rightarrow \quad u_4 - \frac{2}{3} u_4 = 00$

$\Rightarrow \quad u_3 = \frac{2}{3}(0.03)$

$\Rightarrow \quad u_3 = 0.02\, mm$

$\Rightarrow \quad 300 \times 10^3 \left[u_2 - \frac{1}{2} u_3 + 0\ u_4 + 0\, u_5 \right] = 0$

$\Rightarrow \quad u_2 - \frac{1}{2} u_3 = 0$

$\Rightarrow \quad u_2 = \frac{1}{2} u_3 = \frac{1}{2} \times 0.02$

$\Rightarrow \quad u_2 = 0.01\, mm$

Result: Displacement

$u_1 = 0$

$u_2 = 0.01\, mm$

$u_3 = 0.02\, mm$

$u_4 = 0.03\, mm$

$u_5 = 0.04\, mm$

Example 2.10

Consider a bar as shown in Fig. (i). An axial load of 200 kN is applied at point p. Take $A_l = 2400$ mm^2, $E_l = 70 \times 10^9$ N/m^2, $A_2 = 600$ mm^2, $E_2 = 200$ x 10 N/m^2 .

Calculate the following:

(i) The nodal displacement at point P,

(ii) Stress in each material.

(iii) Reaction force

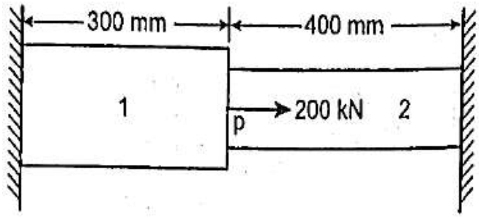


Fig. (i)

Given:

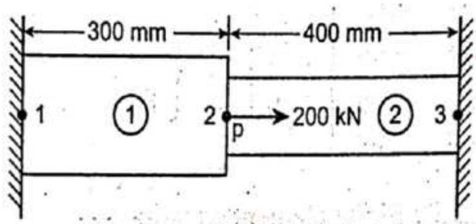


Fig. (ii)

Area of element (1), $A_1 = 2400\ mm^2$

Area of element (2), $A_2 = 600\ mm^2$

Length of element (1), $l_1 = 300$ mm

Length of element (2), $l_2 = 400$ mm

Young's modulus of element (1), $E_1 = 70 \times 10^9\ N/m^2 = 70 \times 10^3\ N/mm^2$

Young's modulus of element (1), $E_2 = 200 \times 10^9\ N/m^2 = 200 \times 10^3\ N/mm^2$

Point load, p = 200 Kn = 200×10^3 N

To find:

(i) Nodal displacement at point p, i.e., u_2

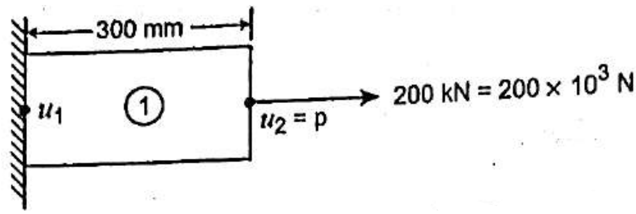
(ii) Stress in each material, σ_1 and σ_2.

(iii)Reaction force, R_1, R_2.

Soltuion: Finite element equaiton for one dimensional two noded bar element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{[From equaiton no. 2.36]}$$

For element 1: (Nodes 1, 2):



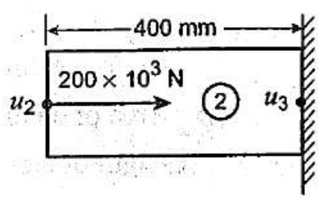
Finite element equation is,

$$\frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \frac{2000 \times 70 \times 10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \begin{bmatrix} a_{11} & a_{12} \\ 5.6 & -5.6 \\ a_{21} & a_{22} \\ -5.6 & 5.6 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1)}$$

For element 2: (Nodes 2, 3): Finite element equation is,



$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \frac{600 \times 200 \times 10^3}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \begin{bmatrix} a_{22} & a_{23} \\ 3 & -3 \\ a_{32} & a_{33} \\ -3 & 3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \text{... (2)}$$

Assemble the finite elements. i.e., assemble the finite element equations (1) and (2).

$$\Rightarrow \qquad 1 \times 10^5 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 5.6 & -5.6 & 0 \\ a_{21} & a_{22} & a_{23} \\ -5.6 & 5.6+3 & -3 \\ a_{31} & a_{32} & a_{33} \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \qquad 1 \times 10^5 \quad \underset{\substack{\downarrow \\ [K]}}{\begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \ldots (3)$$

[Note: The bar has 3 nodes. Each node has single degree of freedom. So, the global stiffness matrix [K] size is 3× 3. The properties of the stiffness matrix are also satisfied.

(i) [K] matrix is symmetric.

(ii) The sum of elements in any column is equal to zero.

Applying boundary conditions:

Displacements at node 1 and node 3 are zero. So, $u - 1 = u_3 = 0. A\ load\ of\ 200\ \times 10^3\ N$ is acting at node 2. So, $F_2 = 200\ \times 10^3 N$. Self-weight is neglected. i.e., $F_1 = F_3 = 0$. Substitute $u_1, u_3\ and\ F_1, F_2\ and\ F_3$ values in equaiton (3).

$$(3) \qquad \Rightarrow 1\ \times 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.8 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{Bmatrix}$$

In the above equation, u1 = 0. So, neglect first row and first column of [K] matrix. $u_3 = 0$, so, neglect third row and third column of [K] matrix. The final reduced equation is,

$$1\ \times 10^5 [8.6]\{\ u_2\} = \{\ 2\ \times 10^5\}$$

$$\Rightarrow \qquad 8.6\ \times 10^5 u_2 = 2\ \times 10^5$$

$$8.6\, u_2 = 2$$

$$u_2 = 0.2325\ mm$$

Stress in each element:

We know that, $Stress, \quad \sigma = E\dfrac{du}{dx}$

For element (1), $Stress, \sigma_1 = E_1 \dfrac{u_2 - u_1}{l_1} = 70 \times 10^3 \times \dfrac{(0.2325 - 0)}{300}$

$$\sigma_1 = 54.25\ N/mm^2$$

For element (2), $Stress, \sigma_2 = E_2 \dfrac{u_3 - u_2}{l_2}$

$$= 200 \times 10^3 \times \frac{(0 - 0.2325)}{400}$$

$$\Rightarrow \qquad \sigma_2 = -116.25\ \frac{N}{mm^2} \text{(Compressive stress is acting)}$$

Reaction force: we know that,

Reaction force, $\{R\} = [K]\{u^*\} - \{F\}$

$$\Rightarrow \qquad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.2325 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 0 - 5.6\,(0.2325) + 0 \\ 0 + 8.6\,(0.2325) + 0 \\ 0 - 3\,(0.2325) + 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{Bmatrix} -1.302 \\ 2 \\ -0.6975 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.302 \times 10^5 \\ 2 \times 10^5 \\ -0.6975 \times 10^5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -1.302 \times 10^5 \\ 0 \\ -0.6975 \times 10^5 \end{Bmatrix}$$

$\Rightarrow \quad R_1 = -1.302 \times 10^5$

$R_2 = 0\,N$

$R_3 = -0.6975 \times 10^5\,N$

We know that, Reaction force is equivalent and opposite to the applied force.

Verification: $R_1 + R_2 + R_3 = -1.302 \times 10^5 - 0 - 0.6975 \times 10^5$

$= -200 \times 10^3 N (Applied\ force)$

Result:

(i) Nodal displacement at point p, i.e., $u_2 = 0.2325$ mm

(ii) Stress in each material,

$\sigma_1 = 54.25$ N/mm^2 (tensile)

$\sigma_2 = -116.25$ N/mm^2 (Compressive)

(iii) Reaction force, $R_1 = -1.302 \times 10^5$

$R_2 = 0$

$R_3 = -0.6975 \times 10^5\,N$

Example 2.11

A thin steel plate of uniform thickness 25 mm is subjected to a point load of 420 N at mid depth as shown in Fig.(i). The plate is also subjected to self-weight. If Young's modulus, E = 2 x 10^5 N/mm2 and unit weight density, $\rho = 0.8 \times 10^{-4}$ N/mm^3, calculate the following: (i) Displacement at each nodal point. (ii) Stresses in each element.

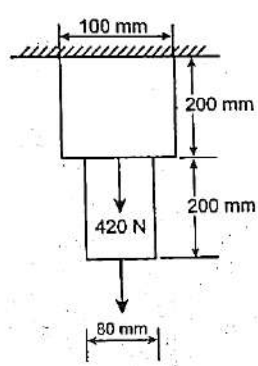


Fig. (i)

Given:

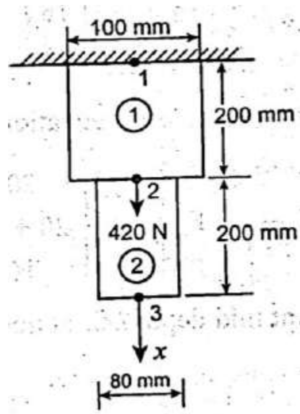


Fig. (ii)

Thickness, t = 25 mm

For element (1): Area, $A_1 = 100 \times 25 = 2500$ mm^2

For element (2): Area, $A_2 = 80 \times 25 = 2000$ mm^2

Point load, p = 420 N

Young's modulus, $E = 2 \times 10^5$ N/mm^2

Unit weight density, $\rho = 0.8 \times 10^{-4}$ N/mm^2

To find: (i) Displacement at each nodal points, $u_1, u_2 and\ u_3$.

(ii) Stress in each element, σ_1 and σ_2.

Solution: The Steel plat is subjected to self-weight. So, we have to find the body force acting at nodal point 1, 2 and 3.

we know thta, body force vector, $\{F\} = \frac{\rho\, A\, l}{2}$ [From equation no. (2.44)]

For element (1): force vector, $\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{\rho_1\, A_1\, l_1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$= \frac{0.8 \times 10^{-4} \times 2500 \times 200}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 20 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix} \qquad ...(1)$$

For element (2): force vector, $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{\rho_2 A_2 l_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$= \frac{0.8 \times 10^{-4} \times 2000 \times 200}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 16 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 16 \\ 16 \end{Bmatrix} \qquad \ldots (2)$$

Assembling the force vector, i.e., assemble the equation (1) and (2)

$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20+16 \\ 16 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 36 \\ 16 \end{Bmatrix}$$

A Point load of 420 N is acting at mid depth i.e., at nodal point 2, as shown in Fig(ii) So, add 420 N in F_2 vector.

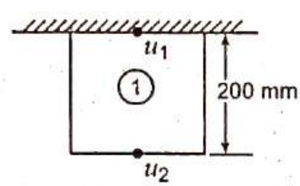
$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 36+420 \\ 16 \end{Bmatrix}$$

Global force vector $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 456 \\ 16 \end{Bmatrix} \qquad \ldots (3)$

Finite element equation for one dimensional plate element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{[From equaiton no. (2.36)]}$$

For element 1: (Nodes 1, 2):



Finite element equation is,

$$\frac{A_1 E}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

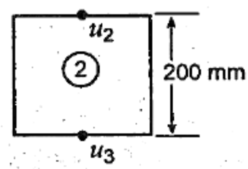
$$\Rightarrow \frac{2500 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad 2 \times 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 12.5 & -12.5 \\ -12.5 & 12.5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (4)}$$

For element 2: (Nodes 2, 3): Finite element equation is,

$$\frac{A_2E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \frac{2000 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



$$\Rightarrow \quad 2 \times 10^5 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 12.5 & -12.5 \\ -12.5 & 12.5 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (5)}$$

Assemble the finite elements i.e., assemble the finite element equations (4) and (5).

$$\Rightarrow \quad 2 \times 10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 12.5 + 10 & -10 \\ 0 & -10 & 10 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \quad 2 \times 10^5 \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 22.5 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \text{.. (6)}$$

Apply the boundary conditions i.e., at node 1, displacement $u_1 = 0$. Substitute u_1, F_1, F_2 and F_3 values in equation (6).

$$\Rightarrow \quad 2 \times 10^5 \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 22.5 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} 20 \\ 456 \\ 16 \end{Bmatrix}$$

In the above equation, $u_1 = 0$. So, neglect first now and first column of [K]matrix. The reduced equation is,

$$\Rightarrow \quad 2 \times 10^5 \begin{bmatrix} 22.5 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 456 \\ 16 \end{Bmatrix}$$

$\Rightarrow$ $$2 \times 10^5(22.5\, u_2 - 10\, u_3) = 456 \qquad ....(7)$$

$$2 \times 10^5(-10\, u_2 + 10\, u_3) = 16 \qquad ....(8)$$

Solving $$2 \times 10^5(12.5\, u_2) = 472$$

$\Rightarrow$ $$u_2 = 1.888 \times 10^4\, mm$$

Substitute u_2 value in equation (7),

$$\Rightarrow 2 \times 10^5[-10(1.888 \times 10^{-4}) + 10\, u_3] = 16$$

$\Rightarrow$ $$-10(1.888 \times 10^{-4}) + 10\, u_3 = 8 \times 10^{-5}$$

$\Rightarrow$ $$-10\, u_3 = 1.968 \times 10^{-5}$$

$$u_3 = 1.968 \times 10^{-4} mm$$

we know that, stress, $$\sigma = E\frac{du}{dx}$$

For element (1): $$\sigma_1 = E \times \frac{u_2 - u_1}{l_1} = 2 \times 10^5 \times \frac{1.888 \times 10^{-4} - 0}{200}$$

$$\sigma_1 = 0.188\, N/mm^2$$

For element (2): $$\sigma_2 = E \times \frac{u_3 - u_2}{l_2}$$

$$= 2 \times 10^5 \times \frac{1.968 \times 10^{-4} - 1.888 \times 10^{-4}}{200}$$

$$\sigma_2 = 0.008 N/mm^2$$

Result: (i) Displacement at each nodal points:

$$u_1 = 0$$

$$u_2 = 1.888 \times 10^{-4}\, mm$$

$$u_3 = 1.968 \times 10^{-4}\, mm$$

(ii) Stresses in each element:

$\sigma_1 = 0.188$ N/mm^2

$\sigma_2 = 0.008$ N/mm^2

Example 2.12

The three bar assemblage is shown in fig (i). a force of 2500 N is appied in the x direction at node 2. The length of each element is 750 mm. Take E = 4 ×10^5 N/mm^2 and A=1200mm^2 for elments 1 and 2.

Take E=2 × 10^5 N/mm^2 and A=1200mm^2 for element 3. Nodes 1 and 4 are fixed.

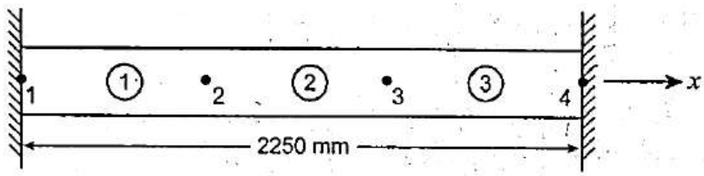


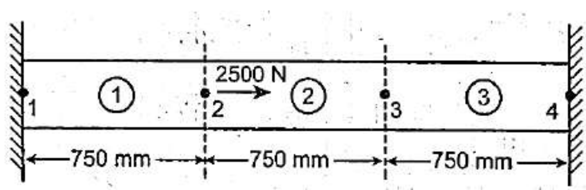
Fig. (i)

Calculate the following: (i) Global stiffness matrix.

(ii) Displacemnets of nodes 2 and 3.

(iii) Reactions at nodes 1 and 4.

Given:



[Note: Number with circle denoted elements & number without circle denotes nodes]

$l_1 = 750\ mm;$ $l_2 = 750\ mm;$ $l_3 = 750\ mm;$

$F_2 = 2500\ N;$ $E_1 = 4\ \times 10^5 N/mm^2 = E_2;$

$A_1 = 600\ mm^2 = A_2;$ $E_3 = 2\ \times 10^5 N/mm^2;$ $A_3 = 1200\ mm^2$

To find: (i) Global stiffness matrix [K].

(ii) Displacemnets of nodes 2 and 3 (u_2, u_3).

(iii) Reactions at nodes 1 and 4 (R_1, R_4).

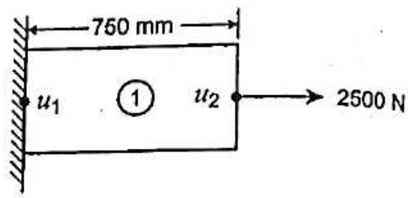
Solution: Finite element equation for one dimensional two noded bar element is given by,.

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

[From equation no (2.36)]

For element 1: (Nodes 1, 2): Finite element equation is,

$$\frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



$$\Rightarrow \quad \frac{600 \times 4 \times 10^5}{750}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

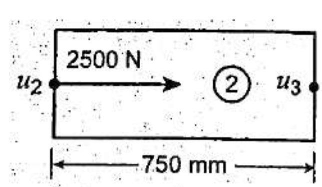
$$\Rightarrow \quad 3.2 \times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \begin{array}{c} \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1)}$$

For element 2: (Nodes 2, 3): Finite element equation is,

$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \quad \frac{600 \times 4 \times 10^5}{7500}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



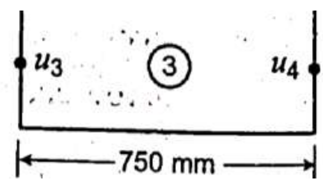
$$\Rightarrow \quad 3.2 \times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow 1 \times 10^5 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \quad \text{... (2)}$$

For element 3: (Nodes 3, 4): Finite element equation is,

$$\frac{A_3 E_3}{l_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\Rightarrow \frac{1200 \times 2 \times 10^5}{750} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$



$$\Rightarrow \quad 3.2 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \end{array} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} \quad \text{... (3)}$$

Assemble the finite elements i.e., assemble the finite element equations (1), (2) and (3).

$$\Rightarrow 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3.2 & -3.2 & 0 & 0 \\ -3.2 & 3.2+3.2 & -3.2 & 0 \\ 0 & -3.2 & 3.2+3.2 & -3.2 \\ 0 & 0 & -3.2 & 3.2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \underbrace{\begin{bmatrix} 3.2 & -3.2 & 0 & 0 \\ -3.2 & 6.4 & -3.2 & 0 \\ 0 & -3.2 & 6.4 & -3.2 \\ 0 & 0 & -3.2 & 3.2 \end{bmatrix}}_{[K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \quad \text{... (4)}$$

[Note: The bar has 4 nodes. Each node has single degree of freedom. So, the global stiffness matrix [K] size is 4×4. The properties of the stiffness matrix are also satisfied.]

Applying boundary conditions:

(i) Displacements at node 1 and node 4 are zero $\Rightarrow u_1 = u_4 = 0$.

(ii) 2500 N load is acting at node 2 $\Rightarrow F_2 = 2500\ N$.

(iii)Self-weight is neglected $\Rightarrow F_1 = F_3 = F_4 = 0$

Substitute $u_1\ , u_4, F_1, F_2, F_3\ , F_4\ values\ in\ equaiton\ (4)$.

$$\Rightarrow 1 \times 10^5 \quad \begin{bmatrix} 3.2 & -3.2 & 0 & 0 \\ -3.2 & 6.4 & -3.2 & 0 \\ 0 & -3.2 & 6.4 & -3.2 \\ 0 & 0 & -3.2 & 3.2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2500 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1 = 0$. So, neglect first now and first column of [K]matrix. $u_4 = 0$, so, neglected fourth row and fourth column of [K] matrix. The final reduced equation is,

$$1 \times 10^5 \begin{bmatrix} 6.4 & -3.2 \\ -3.2 & 6.4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2500 \\ 0 \end{Bmatrix}$$

$$1 \times 10^5 (6.4\, u_2 - 3.2\, u_3) = 2500 \qquad \text{....}(5)$$

$$1 \times 10^5 (-3.2\, u_2 + 6.4\, u_3) = 0 \qquad \text{....}(6)$$

Multiple the equation (6) by 2,

$$1 \times 10^5 (6.4\, u_2 - 3.2\, u_3) = 2500$$

$$1 \times 10^5 (-6.4\, u_2 + 12.8\, u_3) = 0 \qquad \text{....}(7)$$

Solving $\quad 1 \times 10^5 (9.6\, u_3) = 2500$

$\Rightarrow \quad u_3 = 2.604 \times 10^{-3}\ mm$

Substitute u_3 value in equation (5),

$\Rightarrow \quad 1 \times 10^5 [6.5\, u_2\ 3.2(2.604 \times 10^{-3})] = 2500$

$\Rightarrow \quad 6.5\, u_2 - 8.33 \times 10^{-3} = 0.025$

$\Rightarrow \quad u_2 = 5.127 \times 10^{-3}\ mm$

we know that, sReaction force, $\{R\} = [K\ ]\{u^*\} - \{F\}$

$$\Rightarrow \begin{Bmatrix} R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3.2 & -3.2 & 0 & 0 \\ 3.2 & 6.4 & -3.2 & 0 \\ 0 & -3.2 & 6.4 & -3.2 \\ 0 & 0 & 0 & 3.2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3.2 & -3.2 & 0 & 0 \\ 3.2 & 6.4 & -3.2 & 0 \\ 0 & -3.2 & 6.4 & -3.2 \\ 0 & 0 & -3.2 & 3.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 5.127 \times 10^{-3} \\ 2.604 \times 10^{-3} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2500 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 0 - 3.2 \times 5.127 \times 10^{-3} + 0 + 0 \\ 0 + 6.4 \times 5.127 \times 10^{-3} - 3.2 \times 2.604 \times 10^{-3} + 0 \\ 0 + 3.2 \times 5.127 \times 10^{-3} - 6.4 \times 2.604 \times 10^{-3} + 0 \\ 0 + 0 \times 3.2 \times 2.604 \times 10^{-3} + 0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 2500 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{Bmatrix} -0.0164 \\ 0.02448 \\ 2.50 \times 10^{-4} \\ -8.33 \times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2500 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.0164 \times 10^5 \\ 0.02448 \times 10^5 \\ 2.50 \times 10^{-4} \times 10^5 \\ -8.33 \times 10^{-3} \times 10^5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2500 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1640 \\ 2448 \\ 25 \\ -833 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2500 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} -1640 \\ -52 \\ 25 \\ -833 \end{Bmatrix}$$

$\Rightarrow \quad R_1 = -1640\ N; R_2 = -52N; R_3 = 25\ N; R_4 = -833\ N$

We know that, reaction force is equivalent and opposite to the applied force.

Verification: $R_1 + R_2 + R_3 + R_4 = -1640 - 52 + 25 - 833$

= -2500 N (Applies force)

Result: (i) $[K] = 1 \times 10^5 \begin{bmatrix} 3.2 & -3.2 & 0 & 0 \\ 3.2 & 6.4 & -3.2 & 0 \\ 0 & -3.2 & 6.4 & -3.2 \\ 0 & 0 & -3.2 & 3.2 \end{bmatrix}$

(ii) $u_2 = 5.127 \times 10^{-3} mm$

$u_3 = 2.604 \times 10^{-3} mm$

(iii) R_1 = - 1640 N

R_4 = -833 N

[Note: By strictly solving, we will get R_2 =0 and R_3 = 0. But there is some error in the solution due to numeriacal approximation. If we take more number of elements, this error approaches to zero. So, we will get $R_1 + R_4$ = -2500 N (Applied force).

Example 2.13

Consider the bar as shown in fig (i).

Calculate the following: (i) nodal displacement (ii) element stresses, (iii) support reactions

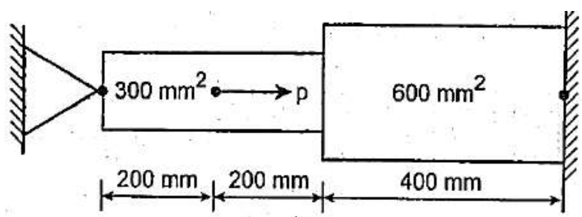


Fig. (i)

Take E = 2×10^5 N/mm^2; p = 400 kN.

Given:

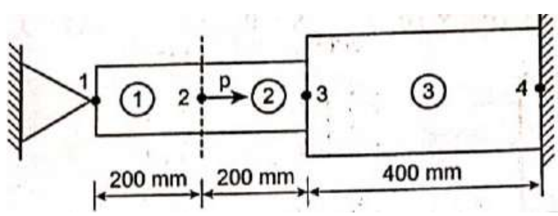


Fig. (ii)

[Note: number with circles denotes elements & Number without circle denotes nodes]

Area of element (1), A_1 = 300 mm^2

Area of element (2), A_2 = 300 mm^2

Area of element (3), A_3 = 600 mm

Length of element (1), $l_1 = 200$ mm

Length of element (2), $l_2 = 200$ mm

Length of element (3), $l_3 = 400$ mm

Young's modulus E = 2×10^5 N/mm^2

Point load at node 2, p = 400 kN = 400×10^3 N

To find:

(i) Nodal displacement u_1, u_2, u_3 *and* u_4.

(ii) Stress in each material, σ_1, σ_2 and σ_3.

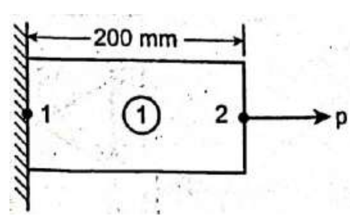
(iii) Reaction at the support, R_1 and R_4.

Soltuion: Finite element equaiton for one dimensional two noded bar element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{[From equaiton no. 2.36]}$$

For element 1: (Nodes 1, 2): Finite element equation is,

$$\frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

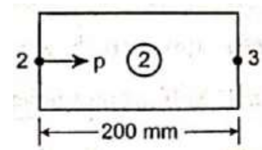


$$\frac{300 \times 2 \times 10^3}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$3 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

For element 2: (Nodes 2, 3): Finite element equation is,



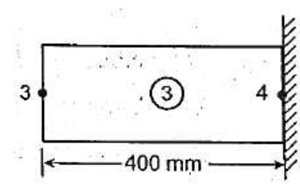
$$\frac{A_2E_2}{l_2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix}$$

$$\frac{300\ \times 2\ \times 10^5}{400}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix}$$

$$3\ \times 10^5\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix}$$

$$\Rightarrow \qquad 1\ \times 10^5\begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}3 & -3\\-3 & 3\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\begin{Bmatrix}F_2\\F_3\end{Bmatrix} \qquad \text{... (2)}$$

For element 3: (Nodes 3, 4): Finite element equation is,



$$\frac{A_3E_3}{l_3}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\F_4\end{Bmatrix}$$

$$\Rightarrow \qquad 3\ \times 10^5\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\F_4\end{Bmatrix}$$

$$\Rightarrow \qquad 1\ \times 10^5\begin{array}{c}\begin{array}{cc}3 & 4\end{array}\\ \begin{bmatrix}3 & -3\\-3 & 3\end{bmatrix}\end{array}\begin{array}{c}3\\4\end{array}\begin{Bmatrix}u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\F_4\end{Bmatrix} \qquad \text{... (3)}$$

Assemble the finite elements. i.e., assemble the finite element equations (1) and (2).

$$\Rightarrow 1\ \times 10^5\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}3 & -3 & 0 & 0\\-3 & 3+3 & -3 & 0\\0 & -3 & 3+3 & -3\\0 & 0 & -3 & 3\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\end{Bmatrix}$$

$$\Rightarrow \qquad 1\ \times 10^5\begin{bmatrix}3 & -3 & 0 & 0\\-3 & 6 & -3 & 0\\0 & -3 & 6 & -3\\0 & 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\end{Bmatrix} \qquad \text{... (4)}$$

Applying boundary conditions:

(i) Node 1 and node 4 are fixed. So, $u_1 = u_4 = 0$.

(ii) $400\ \times 10^3\ N\ is\ acting\ \ at\ node\ 2. so, F_2 = 400\ \times 10^3 N$.

$(ii)\ Self - weight\ is\ neglected. So, F_1 = F_3 = F_4 = 0.$

Substitute u_1, u_4 $and\ F_1, F_2, F_3\ and\ F_4\ values\ in\ equaiton$ (4).

$$(4) \Rightarrow \quad 1 \times 10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0_1 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1 = 0$. So, neglect first row and first column of [K] matrix. $u_4 = 0$, so, delete fourth row and fourth column of [K] matrix. Hence the equation reduced to,

$$1 \times 10^5 \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 400 \times 10^3 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 (6\,u_2 - 3\,u_3) = 400 \times 10^3 \quad \ldots (5)$$

$$1 \times 10^5 (-3\,u_2 + 6\,u_3) = 0 \quad \ldots (6)$$

Multiply the equation (6) by 2.

$$\Rightarrow \quad 1 \times 10^5 (6\,u_2 - 3\,u_3) = 400 \times 10^3 \quad \ldots (5)$$

$$1 \times 10^5 (-6\,u_2 + 12\,u_3) = 0 \quad \ldots (7)$$

Solving, $$1 \times 10^5 (6\,u_3) = 400 \times 10^3$$

$$\Rightarrow \quad u_3 = 0.4444\ mm$$

Substitute u_3 value in equation (5),

$$\Rightarrow \quad 1 \times 10^5\ [6\,u_2 - 3\ (0.4444) = 400 \times 10^3]$$

$$\Rightarrow \quad u_2 = 0.8888\ mm$$

We know that, Stress, $$\sigma = E\frac{du}{dx}$$

For element (1): $Stress,$ $$\sigma_1 = E_1 \frac{u_2 - u_1}{l_1}$$

$$= 2 \times 10^5 \times \frac{(0.8888 - 0)}{200}$$

$$\sigma_1 = 888.88\ N/mm^2 \quad \text{[Tensile stress]}$$

For element (2), : Stress, $$\sigma_2 = E_2 \frac{(u_3 - u_2)}{l_2}$$

$$= 2 \times 10^3 \times \frac{(0.4444 - 0.8888)}{200}$$

$$\sigma_2 = -444.44\ N/mm^2 \quad \text{[Compressive stress]}$$

Reaction force: we know that,

For element (3),: $Stress, \sigma_3 = E_3 \dfrac{(u_4 - u_3)}{l_3} = 2 \times 10^5 \times \dfrac{(0 - 0.4444)}{400}$

$$\sigma_3 = -222.222\ N/mm^2 \quad \text{[Compressive stress]}$$

We know that,

Reaction force, $\{R\} = [K]\{u^*\} - \{F\}$

$$\Rightarrow \qquad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.8888 \\ 0.4444 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{Bmatrix} 0 - 3 \times 0.8888 + 0 + 0 \\ 0 + 6 \times 0.8888 - 3 \times 0.4444 + 0 \\ 0 - 3 \times 0.8888 + 6 \times 0.4444 + 0 \\ 0 + 0 - 3 \times 0.4444 + 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -2.6667 \times 10^5 \\ 4 \times 10^5 \\ 0 \\ -1.3333 \times 10^5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 400 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} -2.6667 \times 10^5 \\ 0 \\ 0 \\ -1.3333 \times 10^5 \end{Bmatrix}$$

$\Rightarrow R_1 = -2.6667 \times 10^5\ N$

$R_2 = 0$

$R_3 = 0$

$R_4 = -1.3333 \times 10^5\ N$

We know that, Reaction force is equivalent and opposite to the applied force.

$$\boldsymbol{Verification}: R_1 + R_2 + R_3 + R_4 = -2.6667 \times 10^5 + 0 + 0 - 1.3333 \times 10^5$$

$$= -4 \times 10^5 N (Applied\ force)$$

Result:

(i) Nodal displacement:

$$u_1 = 0$$

$$u_2 = 0.8888\ mm$$

$$u_3 = 0.4444\ mm$$

$$u_4 = 0$$

(ii) Element Stresses:

σ_1 = 888.88 N/mm^2 (Tensile)

σ_2 = -444.44 N/mm^2 (Compressive)

σ_3 = -222.22 N/mm^2 (Compressive)

(iii) Reaction force, $R_1 = -2.6667 \times 10^5 N$

$R_4 = -1.3333 \times 10^5 N$

Example 2.14

Consider a taper steel plate of uniform thickness, t = 25 mm as shown in Fig. The Young's modulus of the plate, E = 2×10^5 N/mm^2 and weight density, $\rho = 0.82 \times 10^{-4}$ N/mm^3. In addition to its self-weight, the plate is subjected to a point load p = 100 N at its midpoint. Calculate the following by modeling the plate with two finite elements:

(i) Global force vector {F}

(ii) Global stiffness matrix [K]

(iii) Displacements in each element

(iv) Stresses in each element

(v) Reaction force at the support

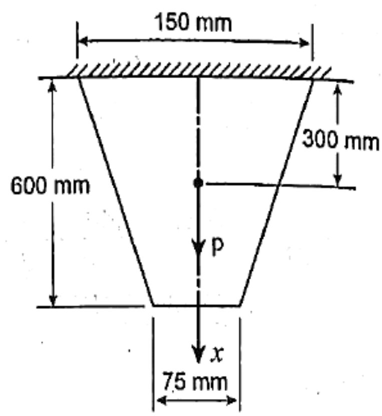


Fig. (i)

Given: In this problem, the area of the element is varying at each cross- section. If we consider this area variation, the problem will be tedious. So, the given taper bar is considered as stepped bas as shown in Fig.(ii).

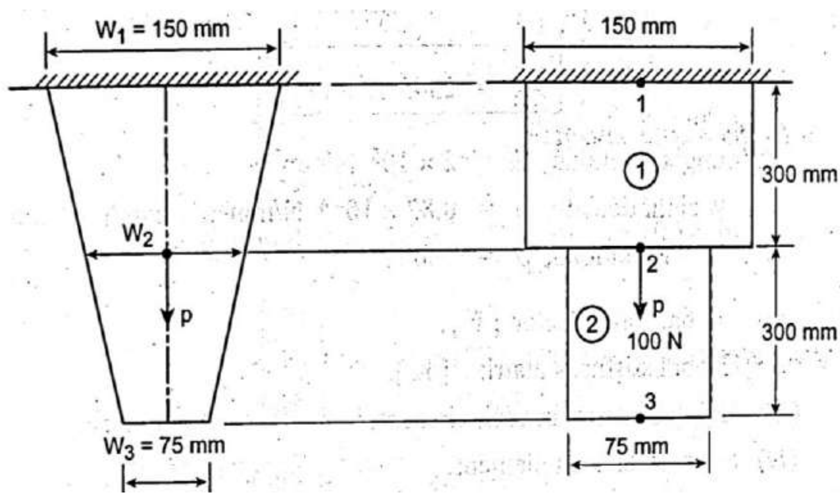


Fig. (ii)

Solution: Area at node 1, $A_1 = \text{Width} \times \text{Thickness} = W_1 \times t_1 = 150 \times 25$

$$A_1 = 3750\ mm^2$$

Area at node 2, $A_2 = W_2 \times t_2$

$$\left(\frac{W_1 + W_3}{2}\right) \times t_2 = \left(\frac{150 + 75}{2}\right) \times 25$$

$$[\because t_1 = t_2 = t_3 = 25\ mm]$$

$$A_2 = 2812.5\ mm^2$$

Area at node 3, $A_3 = W_3 \times t_3 = 75 \times 25$

$$A_3 = 1875\ mm^2$$

Average area of element (1):

$$\overline{A_1} = \frac{Area\ at\ node, 1 + Area\ at\ node, 2}{2}$$

$$= \frac{3750 + 2812.5}{2}$$

$$\overline{A_1} = 3281.25\ mm^2$$

Average area of element (2):

$$\overline{A_2} = \frac{Area\ at\ node, 2 + Area\ at\ node, 3}{2}$$

$$= \frac{2812.5 + 1875}{2}$$

$$\overline{A_2} = 2343.75\ mm^2$$

Young's modulus, E = 2 × 105 N/mm^2

Weight density, *ρ = 0.82 × 10-4 N/mm^3; Length, l = 300 mm*

Point load, p = 100 N

To find: (i) Global force vector {F}.

(ii) Global stiffness matrix, [K].

(iii) Displacement in each element.

(iv) Stresses in each element.

(v) Reaction force at the support.

Solution: the steel plate is subjected to self-weight. So, we have to find the body force acting at nodal points 1, 2 and 3.

We know that,

Body force vector, $\{F\} = \frac{\rho\, A\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ [From equation no. (2.44)]

For element (1), Force vector, $\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \dfrac{\rho\, \bar{A}_1\, l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$= \frac{0.82 \times 10^{-4} \times 3281.25 \times 300}{2} \times \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 40.359 \times \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 40.359 \end{Bmatrix} \qquad \ldots (1)$$

For element (2), Force vector, $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \dfrac{\rho_2\, \bar{A}_2\, l_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$= \frac{0.82 \times 10^{-4} \times 2343.75 \times 300}{2} \times \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 28.828 \times \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 28.828 \\ 28.8289 \end{Bmatrix} \qquad \ldots (2)$$

Assembling the force vector, i.e., assemble the equation (1) and (2),

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 40.359 + 28.828 \\ 28.828 \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 69.187 \\ 28.828 \end{Bmatrix}$$

A point load of 100 N is acting at node 2 as shown in Fig. So, add 100 N in F_2 vector.

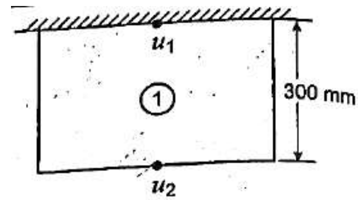
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 69.187 + 100 \\ 28.828 \end{Bmatrix}$$

Golbal force vector, $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix} \qquad \ldots (3)$

Finite element equation for one dimensional plate element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: (Nodes 1, 2): Finite element equation is,



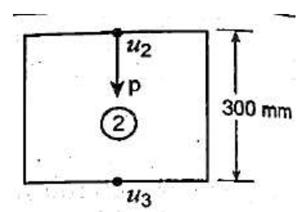
$$\frac{\bar{A}_1 E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\frac{3281.25 \times 2 \times 10^5}{300}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$10.937 \times 2 \times 10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$2 \times 10^5 \begin{array}{c c} \quad 1 & \quad 2 \\ \left[\begin{matrix}10.937 & -10.937\\ -10.937 & 10.937\end{matrix}\right. & \left.\right]\begin{matrix}1\\2\end{matrix}\end{array}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix} \qquad \ldots (4)$$

For element 2: (Nodes 2, 3): Finite element equation is,



$$\frac{\bar{A}_2 E_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

$$\frac{2343.75 \times 2 \times 10^5}{300}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

$$7.8125 \times 2 \times 10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

$$\Rightarrow \quad 2 \times 10^5 \begin{array}{cc} 1 & 2 \\ \multicolumn{2}{c}{\begin{bmatrix}7.8125 & -7.8125\\ -7.8125 & 7.8125\end{bmatrix}} \end{array}\begin{matrix}2\\3\end{matrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix} \qquad \ldots (5)$$

Assemble the finite element equations (4) and (5).

$$\Rightarrow 2 \times 10^5 \begin{array}{ccc} 1 & 2 & 3 \\ \multicolumn{3}{c}{\begin{bmatrix}10.937 & -10.937 & 0\\ -10.937 & 10.937 + 7.8125 & -7.8125\\ 0 & -7.8125 & 7.8125\end{bmatrix}} \end{array}\begin{matrix}1\\2\\3\end{matrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix}$$

$$\Rightarrow \qquad 2 \times 10^5 \underbrace{\begin{bmatrix}10.937 & 10.937 & 0\\ -10.937 & 18.749 & -7.8125\\ 0 & -7.8125 & 7.8125\end{bmatrix}}_{\downarrow\ [K]}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix} \qquad \ldots (6)$$

Applying boundary conditions, i.e., at node 1, displacement, $u_1 = 0$. Substitute F_1, F_2 *and* F_3 in equation (6).

$$\Rightarrow \quad 2\times10^5\begin{bmatrix}10.937 & 10.937 & 0\\ -10.937 & 18.749 & -7.8125\\ 0 & -7.8125 & 7.8125\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}40.359\\ 169.187\\ 28.828\end{Bmatrix}$$

In the above equation, $u_1 = 0$. So, neglect first row and first column of [K] matrix. The reduced equation is,

$$2\times10^5\begin{bmatrix}18.749 & -7.8125\\ -7.815 & 7.8125\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}169.187\\ 28.828\end{Bmatrix}$$

$$\Rightarrow \quad 2\times10^5(18.749\,u_2-7.8125\,u_3)=169.187 \quad \text{... (7)}$$

$$2\times10^5(-7.8125\,u_2+7.8125\,u_3)=28.828 \quad \text{... (8)}$$

Solving, $$2\times10^5(10.936\,)u_2=198.015$$

$$\Rightarrow \quad u_2=9.053\times10^{-5}\,mm$$

Substitute u_2 value in equation (7),

$$\Rightarrow \quad 2\times10^5\,[18.749\,(9.053\times10^{-5})-7.8125\,u_3]=169.187$$

$$\Rightarrow \quad 18.749\times9.053\times10^{-5}-7.8125\,u_3=8.459\times10^{-4}$$

$$-7.8125\,u_3=-8.514\times10^{-4}$$

$$\Rightarrow \quad u_3=10.898\times10^{-5}\,mm$$

We know that, Stress, $\sigma=E\dfrac{du}{dx}$

For element (1): $Stress,\sigma_1=E_1\dfrac{u_2-u_1}{l_1}$

$$=2\times10^5\times\frac{(9.053\times10^{-5}-0)}{300}$$

$$\sigma_1=0.060\,N/mm^2$$

For element (2),: Stress, $\sigma_2=E\times\dfrac{(u_3-u_2)}{l_2}$

$$=\frac{2\times10^5\times(10.8982\times10^{-5}-9.053\times10^{-5})}{300}$$

$$\sigma_2=0.0123\,N/mm^2$$

Reaction force: we know that,

Reaction force, $\{R\} = [K]\{u^*\} - \{F\}$

$$\Rightarrow \quad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 2 \times 10^5 \begin{bmatrix} 10.937 & -10.937 & 0 \\ -10.937 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$= 2 \times 10^5 \begin{bmatrix} 10.937 & -10.937 & 0 \\ -10.937 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix} \begin{Bmatrix} 0 \\ 9.053 \times 10^{-5} \\ 10.898 \times 10^{-5} \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$= 2 \times 10^5 \begin{bmatrix} 0 - 10.937 \times 9.053 \times 10^{-5} + 0 \\ 0 + 18.749 \times 9.053 \times 10^{-5} - 7.8125 \times 10.898 \times 10^{-5} \\ 0 - 7.8125 \times 9.053 \times 10^{-5} + 7.8125 \times 10.898 \times 10^{-5} \end{bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$= 2 \times 10^5 \begin{Bmatrix} -9.901 \times 10^{-4} \\ 8.459 \times 10^{-4} \\ 1.441 \times 10^{-4} \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$= \begin{Bmatrix} -198.02 \\ 169.187 \\ 28.828 \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -238.379 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad R_1 = -2.38.379\ N$$

$$R_2 = 0\ N$$

$$R_3 = 0\ N$$

$$R_4 = -1.3333 \times 10^5\ N$$

We know that, Reaction force is equivalent and opposite to the applied force.

Verification: $\mathrm{R_1 + R_2 + R_3 = -238.379 + 0 + 0 = -238.379\ N}$

Applied force, $\mathrm{F_1 + F_2 + F_3 = 40.359 + 169.187 + 28.828 = 238.37\ N}$

Result:

$$(i)\{F\} = \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$(ii)\ [K] = 2\ \times 10^5 \begin{bmatrix} 10.397 & -10.397 & 0 \\ -10.397 & 18.749 & -7.8125 \\ 0 & -7.8125 & 7.8125 \end{bmatrix}$$

(i) $u_1 = 0$

$u_2 = 9.053 \times 10^{-5}\ mm$

$u_3 = 10.898\ \times 10^{-5}\ mm$

(ii) $\sigma_1 = 0.060$ N/mm^2

$\sigma_2 = 0.0123$ N/mm^2

(iii) Reaction force, $R_1 = \ -238.379N$

$R_2 = 0\ N$

$R_3 = \ 0\ N$

Example 2.15:

For a tapered plate of uniform thickness t = 10 mm as shown in Fig.(i), find the displacements at the nodes by forming into two element model. The bar has mass density, ρ = 7800 kg/m3, Young's modulus, E = 2 $\times 10^5$ MN/m^2. In addition to self- weight, the plate is subjected to a point load p = 10 kN at its Centre. Also determine the reaction force at the support.

[Anna University, ME-Aeronautical Engg-Dec.2006}

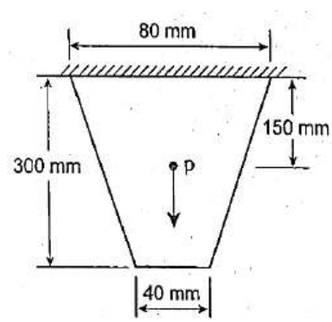


Fig. (i)

Given: In this problem, the area of the element is varying at each cross-section. If we consider this area variation, the problem, will be tedious. So, the given taper bar is considered as stepped bar as shown in Fig. (ii).

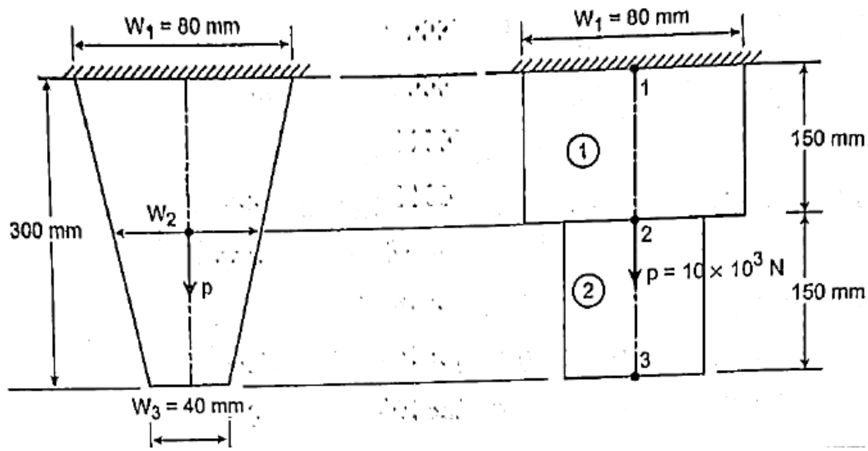


Fig. (ii)

$$\text{Area at node 1}, A_1 = \text{Width } \times \text{Thickness} = W_1 \times t_1$$

$$= 80 \times 10$$

$$A_1 = 800\ mm^2$$

$$\text{Area at node 1}, A_2 = W_2 \times t_2 \left(\frac{W_1 + W_3}{2}\right) \times t_2$$

$$= \left(\frac{80 + 40}{2}\right) \times 10$$

$$A_2 = 600\ mm^2 \qquad [\because t_1 = t_2 = t_3 = 10\ mm]$$

$$\text{Area at node 3}, A_3 = W_3 \times t_3 = 40 \times 10$$

$$A_3 = 400\ mm^2$$

$$\text{Average area of element (1)}, \bar{A}_1 = \frac{\text{Area at nose, 1} + \text{Area at node, 2}}{2}$$

$$\bar{A}_1 = \frac{800 + 600}{2}$$

$$\bar{A}_1 = 700\ mm^2$$

Average area of element (2), $\bar{A}_2 = \dfrac{\text{Area at nose, 2 + Area at node, 3}}{2}$

$$\bar{A}_2 = \frac{600 + 400}{2}$$

$$\bar{A}_2 = 500\ mm^2$$

Mass density, $\rho = 7800\ \text{kg/m}^3$

Weight density, $\rho = 7800\ \times 9.81\ \text{N/m}^3$

$$= 76518\ N/m^3$$

$$= 76518\ \times 10^{-9}\ N/mm^3$$

$$= 7.6518\ \times 10^{-5}\ N/mm^3$$

Young, s modulus, $E = 2\ \times 10^5\ \text{MN/m}^2$

$$= 2\ \times 10^5 \times 10^6\ N/m^2$$

$$= 2\ \times 10^5 \times 10^6 \times 10^{-6}\ N/mm^2$$

$$E\ = 2\ \times 10^5\ N/mm^2$$

Point load, $\text{p} = 10\ \text{kN} = 10\ \times 10^3\text{N}$

To find: (i) Displacement at each nodes.

(ii) Reaction force at the support.

Solution: The plate is subjected to self-weight. So, we have to find the body force acting at nodal points 1, 2, 3.

We know that,

Body force vector, $\{F\} = \dfrac{\rho\, A\, l}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$ [From equation no. (2.44)]

For element (1): Force vector $\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \dfrac{\rho\, \bar{A}_1 l}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$

$$= \frac{7.6518 \times 10^{-5} \times 700 \times 150}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$= 4.017\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 4.017 \end{Bmatrix} \qquad \ldots (1)$$

For element (2): Force vector $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \dfrac{\rho \bar{A}_2 l_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$= \frac{7.6518 \times 10^{-5} \times 500 \times 150}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 2.869 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 2.869 \\ 2.869 \end{Bmatrix} \qquad \ldots (2)$$

Assembling the force vector, i.e., assemble the equation (1) and (2),

$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 4.017 + 2.869 \\ 2.869 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 6.886 \\ 2.869 \end{Bmatrix}$$

A point load of 10×10^3 N is acting at node 2 as shown in Fig. So, add 10,000 N in F2 vector.

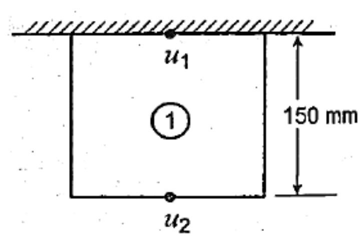
$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 6.886 + 10{,}000 \\ 2.869 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 10006.886 \\ 2.869 \end{Bmatrix}$$

Global force vector, $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 10006.886 \\ 2.869 \end{Bmatrix} \qquad ..(3)$

Finite element equation for one dimensional plate element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: (Nodes 1,2): Finite element equation is,



$$\frac{\bar{A}_1 E}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

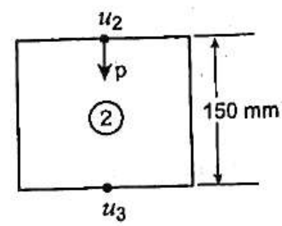
$$\frac{700 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$4.666 \times 2 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$2 \times 10^5 \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \dots (4)$$

For element 2: (Nodes 2, 3): Finite element equation is,

$$\frac{\bar{A}_2 E}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



$$\Rightarrow \quad \frac{500 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \; 3.333 \times 2 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ 3 \end{Bmatrix}$$

$$\Rightarrow \quad 2 \times 10^5 \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \dots (5)$$

Assemble the finite element equations (4) and (5).

$$\Rightarrow 2 \times 10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4.666 & -4.666 & 0 \\ -4.666 & 4.666 + 3.333 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \quad 2 \times 10^5 \underbrace{\begin{bmatrix} 4.666 & -4.666 & 0 \\ -4.666 & 7.999 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix}}_{[K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \dots (6)$$

Apply the boundary conditions, i.e., at nose 1, displacement $u_1 = 0$. Substituting u_1, F_1, F_2 and F_3 values in equation (6),

$$\Rightarrow 2 \times 10^5 \begin{bmatrix} 4.666 & -4.666 & 0 \\ -4.666 & 7.999 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 10006.886 \\ 2.869 \end{Bmatrix}$$

In the above equation u1= 0. So, neglect first row and first column of [K] matrix. The reduced equation is,

$$2 \times 10^5 \begin{bmatrix} 7.999 & -3.333 \\ -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 10{,}006.886 \\ 2.869 \end{Bmatrix}$$

$$\Rightarrow \quad 2 \times 10^5 (7.999\, u_2 - 3.333\, u_3) = 10{,}006.886 \quad \dots (7)$$

$$2 \times 10^5(-3.333\, u_2 + 3.333\, u_3) = 2.869 \qquad \ldots (8)$$

Solving, $\quad 2 \times 10^5(4.666\, u_2) = 10009.755$

$\Rightarrow \quad u_2 = 0.01073\, mm$

Substitute u_2 value in equation (8),

$\Rightarrow \quad 2 \times 10^5\, [-3.333(0.01073) + 3.333\, u_3] = 2.869$

$\Rightarrow \quad -3.333(0.01073) + 3.333\, u_3 = 1.4345 \times 10^{-5}$

$\Rightarrow \quad u_3 = 0.01073\, mm$

We know that, Reaction force, $\{R\} = [K]\{u^*\} - \{F\}$

$$\Rightarrow \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 2 \times 10^5 \begin{bmatrix} 4.666 & -4.666 & 0 \\ -4.666 & 7.999 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$= 2 \times 10^5 \begin{bmatrix} 4.666 & -4.666 & 0 \\ -4.666 & 7.999 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.01073 \\ 0.01073 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10{,}006.886 \\ 2.869 \end{Bmatrix}$$

$$= 2 \times 10^5 \begin{bmatrix} 0 + 4.666 \times 0.01073 + 0 \\ 0 + 7.999 \times 0.01073 - 3.333 \times 0.01073 \\ 0 - 3.333 \times 0.01073 + 3.333 \times 0.01073 \end{bmatrix} - \begin{Bmatrix} 4.017 \\ 10{,}006.886 \\ 2.869 \end{Bmatrix}$$

$$= 2 \times 10^5 \begin{Bmatrix} -0.050 \\ 0.050 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10{,}006.886 \\ 2.869 \end{Bmatrix}$$

$$= \begin{Bmatrix} -10{,}000 \\ 10{,}000 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 10{,}006.886 \\ 2.869 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -10004.017 \\ -6.886 \\ -2.869 \end{Bmatrix}$$

$\Rightarrow R_1 = -10004.07\, N$

$R_2 = -6.886\, N$

$R_3 = -2.869\, N$

We know that, Reaction force is equivalent and opposite to the applied force.

$\boldsymbol{Verification}$: $R_1 + R_2 + R_3 = -10004.017 + 6.886 - 2.869$

$$= -10013.772\ N$$

$$\textbf{Applied force}, F_1 + F_2 + F_3 = 4.017 + 10{,}006.886 + 2.869$$

$$= 10013.772\ N$$

Result:

(i) Displacement at each node:

$$u_1 = 0$$

$$u_2 = 0.01073\ mm$$

$$u_3 = 0.01073\ mm$$

(ii) Reaction force at the support:

$$R_1 = -10004.017\ N$$

Example 2.16

Using two finite elements, find the stress distribution in a uniformly tapering bar of circular cross-sectional area 3 cm^2 and 2cm^2 at their ends, length 100mm, subjected to an axial tensile load of 50 N at smaller end and fixed at larger end Take the value of young's modulus $2*10^5$ N/mm^2. [Anna university ME.- jan,2006]

Given:

In this problem, the area of the element is varying at each cross-section. If we consider this area variation, the problem will be tedious. So, the given taper bar is considered as stepped bar as shown in Fig.(i).

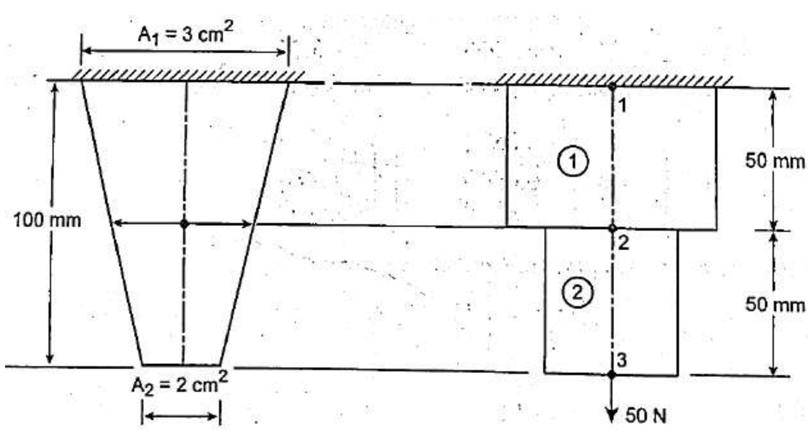


Fig. (i)

Area at node 1, $A_1 = 3 \text{ cm}^2$

Area at node 3, $A_3 = 2 \text{ cm}^2$

Area at node 2, $A_2 = \frac{A_1 + A_3}{2} = \frac{3+2}{2} = 2.5 \text{ cm}^2$

Average area of element (1), $\bar{A}_1 = \frac{\text{Area at nose, 1 + Area at node, 2}}{2}$

$$\bar{A}_1 = \frac{3 + 2.5}{2}$$

$$\bar{A}_1 = 2.75\, cm^2 = 2.75 \times 10^2\, mm^2$$

$$\bar{A}_1 = 275\, mm^2$$

Average area of element (2),

$$\bar{A}_2 = \frac{\text{Area at nose, 2 + Area at node, 3}}{2}$$

$$= \frac{2.5 + 2}{2} = 2.25 \text{ cm}^2$$

$$\bar{A}_2 = 225 \text{ mm}^2$$

Tensile load at node, 3 = 50 N

Young's modulus, $E = 5 \times 10^5 \text{ N/mm}^2$

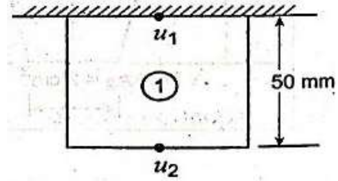
To find: Stress distribution: (i) Stress in element (1), σ_1.

(ii) Stress in element (2), σ_2.

Solution: Finite element equation for one dimensional two noded bar element is given by,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: (Nodes 1, 2): Finite element equation is,

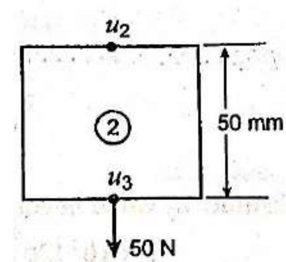


$$\frac{\bar{A}_1\, E}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{275 \times 2 \times 10^5}{50} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow 2 \times 10^5 \begin{array}{cc} \begin{array}{cc} 1 & \quad 2 \end{array} & \\ \begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1)}$$

For element 2: (Nodes 2, 3): Finite element equation is,



$$\frac{\bar{A}_2 E}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\frac{225 \times 2 \times 10^5}{50} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$1 \times 10^5 \begin{array}{cc} \begin{array}{cc} 2 & \quad 3 \end{array} & \\ \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \text{... (2)}$$

Assemble the finite element equations (1) and (2).

$$\Rightarrow 1 \times 10^5 \begin{array}{cc} \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} & \\ \begin{bmatrix} 11 & -11 & 0 \\ -11 & 11+9 & -9 \\ 0 & -9 & 9 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow 1 \times 10^5 \underbrace{\begin{bmatrix} 11 & -11 & 0 \\ -11 & 11+9 & -9 \\ 0 & -9 & 9 \end{bmatrix}}_{[K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \text{... (3)}$$

Apply boundary conditions:

(i) At nose 1, displacement $u_1 = 0$.

(ii) Self-weight is neglected and 50 N is acting at node 3. So, $F_1 = 0$, $F_2 = 0$, $F_3 = 50\ N$. Substituting u_1, F_1, F_2 and F_3 values in equation (3),

$$\Rightarrow 1 \times 10^5 \begin{bmatrix} 11 & -11 & 0 \\ -11 & 20 & -9 \\ 0 & -9 & 9 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

In the above equation $u_1 = 0$. So, neglect first row and first column of [K] matrix. Hence, the equation reduces to,

$$1 \times 10^5 \begin{bmatrix} 20 & -9 \\ -9 & 9 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 50 \end{Bmatrix}$$

$$\Rightarrow \qquad 1 \times 10^5(20\,u_2 - 9\,u_3) = 0 \qquad \dots(4)$$

$$1 \times 10^5(-9\,u_2 + 9\,u_3) = 50 \qquad \dots(5)$$

Solving, $\quad 1 \times 10^5(11\,u_2) = 50$

$$\Rightarrow \qquad u_2 = 4.545 \times 10^{-5}\,mm$$

Substitute u_2 value in equation (4),

$$\Rightarrow 1 \times 10^5\,[20\;(4.545 \times 10^{-5}) - 9\,u_3] = 0$$

$$\Rightarrow \qquad 20(4.545 \times 10^{-5}) = 9\,u_3$$

$$\Rightarrow \qquad u_3 = 1.01 \times 10^{-4}\,mm$$

We know that, $Stress \qquad \sigma = E\,\dfrac{du}{dx}$

For element (1): $Stress\;\sigma_1 = E\,\dfrac{u_2 - u_1}{l_1}$

$$= E\,\frac{u_2 - u_1}{l_1}$$

$$= \frac{2 \times 10^5(4.545 \times 10^{-5} - 0)}{50}$$

$$\sigma_1 = 0.1818\,N/mm^2$$

For element (2): Stress $\sigma_2 = E \times \dfrac{u_3 - u_2}{l_2}$

$$= \frac{2 \times 10^5(1.01 \times 10^{-4} - 4.545 \times 10^{-5})}{50}$$

$$\sigma_2 = 0.222\,N/mm^2$$

Verification: *we know that,*

$$\text{Stress}, \sigma = \frac{\text{Load}}{\text{Area}}$$

For element (1): $\sigma_1 = \dfrac{\text{P}}{\bar{\text{A}}_1} = \dfrac{50}{275}$

$$\sigma_1 = 0.1818\,N/mm^2$$

For element (2): $\sigma_2 = \dfrac{P}{\overline{A}_2} = \dfrac{50}{225}$

$$\sigma_2 = 0.222\ N/mm^2$$

Example 2.17

A rod subjected to an axial load P = 600 KN is applied as shown in Fig. Divide the domain into two elements. Determine the following:

(a) Displacement at each node.

(b) Stresses in each element.

(c) Reactions at each node point.

Take A = 250 mm2, E = 2 x 10^5 N/mm^2 [AU. Dec. 2005]

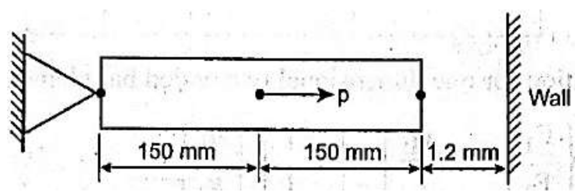


Fig. (i)

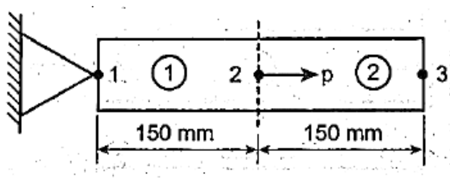


Fig. (ii)

In this problem, we should first determine whether contact occurs between the bar and the wall. To do this, assume that the wall does not exist. The deformation at node is given by,

$$\delta L = \frac{PL}{AE} = \frac{600 \times 10^3 \times 150}{250\ \times 2\ \times 10^5}$$

$$\delta L_{at\ node\ 3} = 1.8\ mm$$

The gap between the wall and node 3 is 1.2 mm. So, the contact occurs between the bar and the wall.

$\Rightarrow$ Displacement, $u_3 = 1.2\ mm$

$\text{Axial load, P} = 600\ \text{kN} = 600\ \times 10^3\text{N}$

$\text{Area, A} = 250\ \text{mm}^2$

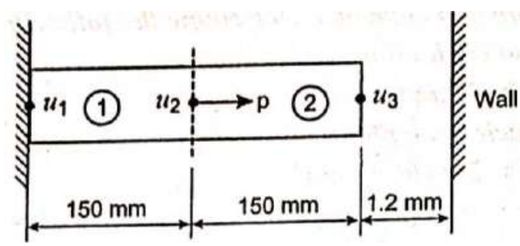
$\text{Young's modulus , E} = 2\ \times 10^5\ \text{mm}^2$

To find: (a) Displacement at each node, u_1, u_2 and u_3.

(b) Stresses in each element, σ_1 and σ_2.

(c) reactions at each node point, R_1, R_2 and R_3.

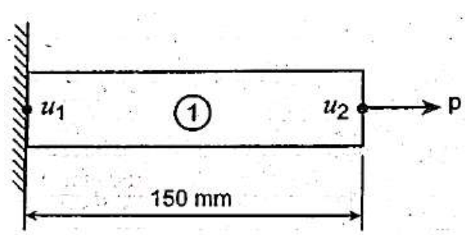
Solution:



Finite element equation for one dimensional two noded bar element is given by,

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For element 1: (Nodes 1, 2):



Finite element equation is,

$$\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

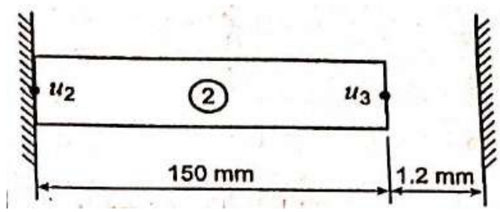
$$\frac{250 \times 2\ \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$3.333\ \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ 1 \times 10^5 & \begin{bmatrix} 3.333 & -3.333 \\ -3.333 & 3.333 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \ldots (1)$$

For element 2: (Nodes 2, 3): Finite element equation is,

$$\frac{A\,E}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



$$\Rightarrow \qquad \frac{250 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ 1 \times 10^5 & \begin{bmatrix} 3.333 & -3.333 \\ -3.333 & 3.333 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \quad \ldots (2)$$

Assemble the finite element equations (1) and (2).

$$\Rightarrow \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ 1 \times 10^5 & \begin{bmatrix} 3.333 & -3.333 & 0 \\ -3.333 & 3.333 + 3.333 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \qquad 1 \times 10^5 \underbrace{\begin{bmatrix} 3.333 & -3.333 & 0 \\ -3.333 & 6.666 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix}}_{\downarrow \atop [K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \ldots (3)$$

Apply boundary conditions:

(i) At nose 1, displacement $u_1 = 0$.

(ii) At node 3, $u_3 = 1.2$ mm (given).

(iii)Point load, $= 600 \times 10^3$ N is acting at node 2 and Self-weight is neglected.

So, $F_1 = 0\,, F_2 = 600 \times 10^3 N\,, F_3 = 0$.

Apply u_1, u_3, F_1, F_2 and F_3 values in equation (3),

$$\Rightarrow \quad 1 \times 10^5 \begin{bmatrix} 3.333 & -3.333 & 0 \\ -3.333 & 6.666 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 1.2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 6 \times 10^5 \\ 0 \end{Bmatrix}$$

In the above equation $u_1 = 0$. So, neglect first row and first column of [K] matrix. Hence, the equation reduces to,

$$1 \times 10^5 \begin{bmatrix} 6.666 & -3.333 \\ -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} u_2 \\ 1.2 \end{Bmatrix} = \begin{Bmatrix} 6 \times 10^5 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 [6.666\, u_2 - 3.333(1.2)] = 6 \times 10^5$$

$$\Rightarrow \quad 6.666\, u_2 - 3.9996 = 6$$

$$\Rightarrow \quad u_2 = 1.50\ mm$$

We know that,

$$Stress\ \sigma = E.\frac{du}{dx}$$

$$\textbf{\textit{For element}}\ (\mathbf{1}): Stress\ \sigma_1 = \frac{E(u_2 - u_1)}{l_1}$$

$$= \frac{2 \times 10^5 (1.5 - 0)}{150}$$

$$\sigma_1 = 2000\ N/mm^2 \qquad [Tensile\ stress]$$

$$\textbf{\textit{For element}}\ (\mathbf{2}): \text{Stress}\ \sigma_2 = \frac{E(u_3 - u_2)}{l_2}$$

$$= \frac{2 \times 10^5 (1.2 - 1.5)}{150}$$

$$= -400\ N/mm^2 \qquad \text{[Compressive stress]}$$

We know that,

Reaction force, $\{R\} = [K]\{u^*\} - \{F\}$

$$\Rightarrow \quad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3.333 & -3.333 & 0 \\ -3.333 & 6.666 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 3.333 & -3.333 & 0 \\ -3.333 & 6.666 & -3.333 \\ 0 & -3.333 & 3.333 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.50 \\ 1.2 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 6 \times 10^5 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{bmatrix} 0 + 3.333(1.50) + 0 \\ 0 + 6.6666\,(1.50) - 3.3333\,(1.2) \\ 0 - 3.333(1.50) - 3.3333\,(1.2) \end{bmatrix} - \begin{Bmatrix} 0 \\ 6 \times 10^5 \\ 0 \end{Bmatrix}$$

$$= 1 \times 10^5 \begin{Bmatrix} -5 \\ 6 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 6 \times 10^5 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -5 \times 10^5 \\ 6 \times 10^5 \\ -1 \times 10^5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 6 \times 10^5 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -5 \times 10^5 \\ 0 \\ -1 \times 10^5 \end{Bmatrix}$$

$\Rightarrow R_1 = -5 \times 10^5\ N = -500\ kN$

$R_2 = 0$

$R_3 = -1 \times 10^5 N = -100\ kN$

We know that, reaction force is equivalent and opposite to the applied force.

Verification: $\mathrm{R_1 + R_2 + R_3 = -500\ kN + 0 - 100\ kN,}$

$$= -600\ kN\ [Applied\ force]$$

Result: $(a). u_1 = 0$

$u_2 = 1.5\ mm$

$u_3 = 1.2\ mm$

$(b).$ $\sigma_1 = 2000\ N/mm^2\ [Tensile]$

$\sigma_2 = -400\ N/mm^2\ [compressive]$

$(c).$ $R_1 = -500\ kN$

$R_2 = 0\ kN$

$R_3 = -100\ kN$

2.10. SPRINGS

Consider a spring element with nodes 1, 2 as shown in Fig.2.2S. Let k be the spring constant and I be the length of the spring. F1 and F2 are the nodal forces acting at node1 and 2 respectively. u_1 and u2 are the displacements at the respective nodes.

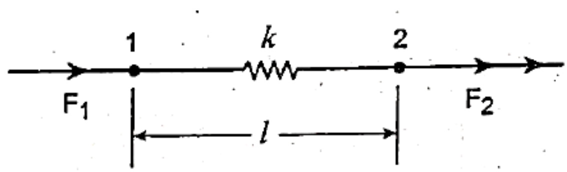


Fig. 2.25.

By the sign convention for nodal forces and equilibrium, we have

$$F_1 = -T_1 \; and \; F_2 = T_2$$

Where T → Tensile force.

We know that,

Tensile force, T = k Δu

Where, k → spring constant, N/m

Δu → Change in deformation, m.

⇒ $F_1 = -\, k\, \Delta u$

$$= k(u_2 - u_1)$$

$$F_1 = k(u_1 - u_2) \qquad \ldots (2.76)$$

⇒ $F_2 = -\, k\, \Delta u$

$$F_2 = k(u_2 - u_1) \qquad \ldots (2.77)$$

Arranging equation (2.76) and (2.77) in matrix form,

$$\Rightarrow \quad \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \ldots (2.78)$$

This is a finite element equation for spring element.

2.10.1 Solved Problems

Example 2.18

A spring assemblage with arbitrarily numbered nodes are shown in Fig. The nodes 1 and 2 are fixed and a force of 500 kN is applied at node 4 ill the x direction. Calculate the following:

(i) Global stiffness matrix.

(ii) Nodal displacements.

(iii) Reactions at each nodal point.

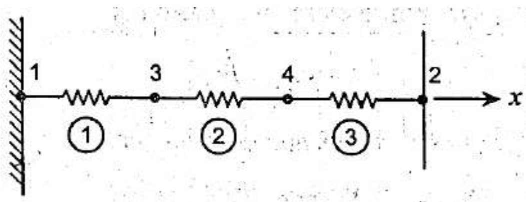


Fig. (i)

$Take\ Spring\ constant, k_1 = 100\ kN/m; k_2 = 200\ kN/m; k_3 = 300\ kN/m;$

Given:

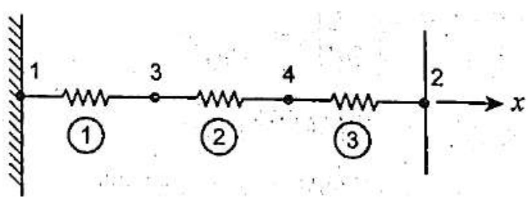


Fig. (ii)

Nodal force, $F_4 = 500$ kN

Spring constant, $k_1 = 100$ kN/m

$k_2 = 200$ kN/m

$k_3 = 300$ kN/m

To find: (i) Global stiffness matrix [K].

(ii) Nodal displacements, u_1, u_2, u_3 and u_4.

(iii) Reactions at each nodal point, R_1, R_2, R_3 and R_4.

Solution: we know that,

Finite element equation for spring element is

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{... [From equation no. (2.78)]}$$

For element 1: (Nodes 1, 3): Finite element equation is,

$$k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \qquad 100 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{array}{cc} 1 & 3 \\ \end{array}$$

$$\begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1)}$$

For element 2: (Nodes 3, 4): Finite element equation is,

$$k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$200 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\begin{array}{cc} 3 & 4 \\ \end{array}$$

$$\Rightarrow \qquad \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} \qquad \text{... (2)}$$

For element 3: (Nodes 4, 2): Finite element equation is,

$$k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_2 \end{Bmatrix}$$

$$300 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_2 \end{Bmatrix}$$

$$\begin{array}{cc} 4 & 2 \\ \end{array}$$

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{matrix} 4 \\ 2 \end{matrix} \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_4 \\ F_2 \end{Bmatrix} \qquad \text{... (3)}$$

Assemble equations (1), (2) and (3)

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ & \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 200+100 & -200 \\ 0 & -300 & -200 & 200+300 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix}}_{\downarrow \atop [K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \qquad \ldots (4)$$

Applying boundary conditions:

At node 1, $u_1 = 0$

At node 2, $u_2 = 0$

Nodal forces, $F_1 = F_2 = F_3 = 0$

$F_4 = 500\ kN\ [Given]$

Substitute F_1, F_2, F_3 *and* $F_{4.}$ u_1 *and* u_2 values in equation (4).

$$\Rightarrow \quad 2 \times 10^5 \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \end{Bmatrix}$$

In the above equation, $u_1 = 0$. So, neglect first row and first column of [K] matrix. $u_2 = 0$, so, neglect second row and second column of [K] matrix. Hence, the equation reduces to

$$\begin{bmatrix} 300 & -200 \\ -200 & 500 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 500 \end{Bmatrix}$$

$$300\, u_3 - 200\, u_4 = 0 \qquad \ldots (5)$$

$$-200\, u_3 + 500\, u_4 = 500 \qquad \ldots (6)$$

$Equation\ (5) \times 2 \Rightarrow \quad 600\, u_3 - 400\, u_4 = 0$

$Equation\ (6) \times 3 \Rightarrow -600\, u_3 + 1500\, u_4 = 1500$

Solving, $\quad 1100\, u_4 = 1500$

$$u_2 = \frac{1500}{1100}$$

$$u_4 = 1.364\ m$$

Substitute u_4 value in equation (5),

$$\Rightarrow \quad 300\ u_3 - 200(1.364) = 0$$

$$\Rightarrow \quad u_3 = 0.9091\ m$$

We know that,

Reaction force, $\{R\} = [K]\{u^*\} - \{F\}$

$$\Rightarrow \quad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$= \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.9091 \\ 1.364 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 + 0 - 100(0.9091) + 0 \\ 0 + 0 + 0 - 300(1.364) \\ 0 + 0 + 300(0.9091) - 200(1.364) \\ 0 + 0 + 200(0.9091) + 500(1.364) \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

$$= \begin{Bmatrix} -90.91 \\ -409.20 \\ 0 \\ 500 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} -90.91 \\ -409.20 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow R_1 = -90.91\ kN$$

$$R_2 = -409.20\ kN$$

$$R_3 = 0$$

$$R_4 = 0$$

Result:

(i) $$[K] = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix}$$

(ii) $u_1 = u_2 = 0$

$u_3 = 0.9091\ m,$

$u_4 = 1.364\ m$

(iii) $R_1 = -90.91\ kN$

$R_2 = -409.20\ kN$

$R_3 = 0$

$R_4 = 0$

Example 2.19

For the bar assemblages shown in fig .(i), determine the nodal displacements, the forces in each element and the reactions

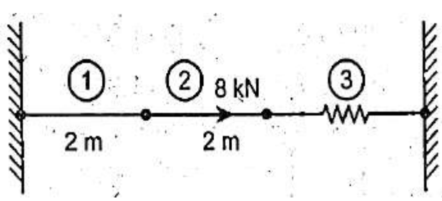


Fig. (i)

E = 70 GPa, A = 2 × 10^{-4} m^2, k = 2000 kN/m

Given:

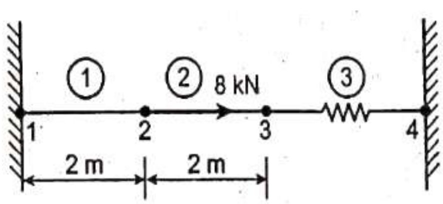


Fig. (ii)

Point load, P = 8kN

Young's modules, $E = 70 \text{ GPa} = 70 \times 10^9 \text{ Pa}$

$= 70 \times 10^9 \text{ N/m}^2$

$= 70 \times 10^6 \text{ kN/m}^2$

Area, $A = 2 \times 10^{-4} \text{ m}^2$

Spring constant, $k = 2000 \text{ kN/m}$

Length of the bar, $l_1 = 2 \text{ m}$

$l_2 = 2 \text{ m}$

To find:

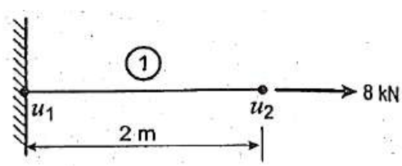
(i) Nodal displacements, u_1, u_2, u_3 and u_4

(ii) Nodal forces in each element.

(iii) Nodal Reactions in each element R_1, R_2, R_3 and R_4

Solution: Finite element equation for one dimensional two noded bar element is given by

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



For element 1: (Nodes 1, 2):

Finite element equation is,

$$\frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \frac{2 \times 10^{-4} \times 70 \times 10^{-6}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

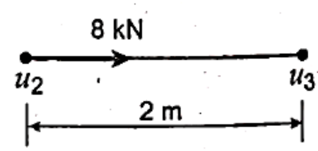
$$\Rightarrow \quad 7 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \qquad 10^3 \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \begin{array}{c}\\\end{array}$$

$$\Rightarrow \qquad 10^3\begin{array}{c} \begin{array}{cc}\mathbf{1} & \mathbf{2}\end{array} \\ \begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1)}$$

For element 2: (Nodes 2, 3): Finite element equation is,

$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$



$$\Rightarrow \frac{2 \times 10^{-4} \times 70 \times 10^{-6}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \qquad 7 \times 10^3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \qquad 10^3\begin{array}{c} \begin{array}{cc}\mathbf{2} & \mathbf{3}\end{array} \\ \begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \text{... (2)}$$

For element 3: (Nodes 3, 4): Finite element equation for spring element is given by,

$$\begin{Bmatrix} \mathrm{F}_3 \\ \mathrm{F}_4 \end{Bmatrix} = \mathrm{k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathrm{u}_3 \\ \mathrm{u}_4 \end{Bmatrix} \quad \text{[From equation no. 2.78)]}$$

$$\Rightarrow \qquad k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathrm{u}_3 \\ \mathrm{u}_4 \end{Bmatrix} = \begin{Bmatrix} \mathrm{F}_3 \\ \mathrm{F}_4 \end{Bmatrix}$$

$$\Rightarrow \qquad 2000\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathrm{u}_3 \\ \mathrm{u}_4 \end{Bmatrix} = \begin{Bmatrix} \mathrm{F}_3 \\ \mathrm{F}_4 \end{Bmatrix}$$

$$\Rightarrow \qquad 10^3\begin{array}{c} \begin{array}{cc}\mathbf{3} & \mathbf{4}\end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{3} \\ \mathbf{4} \end{array} \begin{Bmatrix} \mathrm{u}_3 \\ \mathrm{u}_4 \end{Bmatrix} = \begin{Bmatrix} \mathrm{F}_3 \\ \mathrm{F}_4 \end{Bmatrix} \qquad \text{... (3)}$$

Assemble the equation (1), (2) and (3),

$$\Rightarrow \qquad 10^3 \begin{array}{c} \begin{array}{cccc}\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4}\end{array} \\ \begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 7+7 & -7 & 0 \\ 0 & -7 & 7+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\Rightarrow \qquad 10^3 \begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 14 & -7 & 0 \\ 0 & -7 & 9 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \qquad \text{... (4)}$$

Applying boundary conditions:

Ad node 1, $u_1 = 0$

Ad node 4, $u_4 = 0$

Nodal force $F_1 = F_2 = F_3 = 0$

$F_2 = 8$ kN [Given]

Substitute u_1, u_4, F_2, F_3 and F_4 values in equation (4)

$$\Rightarrow \qquad 10^3 \underbrace{\begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 14 & -7 & 0 \\ 0 & -7 & 9 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}}_{[K]} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 8 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1 = 0$. So, delete first row and first column of [K] matrix. $u_4 = 0$, so, delete fourth row and fourth column of [K] matrix. Hence the equation reduces to,

$$10^3 \begin{bmatrix} 14 & -7 \\ -7 & 9 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \qquad 10^3\,(14u_2 - 7u_3) = 8 \qquad \text{... (5)}$$

$$\Rightarrow \qquad 10^3\,(-7u_2 + 9u_3) = 0 \qquad \text{... (6)}$$

$$10^3\,(14u_2 - 7u_3) = 8$$

$Equation$ (6) × 2 ⇒ $10^3\,(-14u_2 + 18u_3) = 0$

Solving, $\quad 10^3\,(11u_3) = 8$

$$\Rightarrow \qquad u_3 = \frac{8}{11 \times 10^3}$$

$$\Rightarrow \qquad u_3 = 0.727 \times 10^{-3} m$$

Substitute u_3 value in equation (6),

$$\Rightarrow \quad 10^3\,(-7u_2 + 9 \times 0.727 \times 10^{-3}) = 0$$

$$\Rightarrow \qquad u_2 = 0.935 \times 10^{-3} m$$

We know that, Reaction force, {R} = [K] { u*} – {F}

$$\Rightarrow \quad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 10^3 \begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 14 & -7 & 0 \\ 0 & -7 & 9 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$= 10^3 \begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 14 & -7 & 0 \\ 0 & -7 & 9 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.935 \times 10^{-3} \\ 0.727 \times 10^{-3} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 8 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 10^3 \begin{bmatrix} 0 - 7 \times 0.935 \times 10^{-3} + 0 + 0 \\ 0 + 14 \times 0.935 \times 10^{-3} - 7 \times 0.727 \times 10^{-3} + 0 \\ 0 - 7 \times 0.935 \times 10^{-3} + 9 \times 0.727 \times 10^{-3} + 0 \\ 0 + 0 - 2 \times 0.727 \times 10^{-3} + 0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 8 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 10^3 \begin{Bmatrix} -6.546 \times 10^{-3} \\ 8 \times 10^{-3} \\ 0 \\ -1.455 \times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 8 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -6.546 \\ 8 \\ 0 \\ -1.455 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 8 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} -6.546 \\ 8 \\ 0 \\ -1.455 \end{Bmatrix}$$

$$\Rightarrow \quad R_1 = -6546\ kN$$

$$R_2 = 0$$

$$R_3 = 0$$

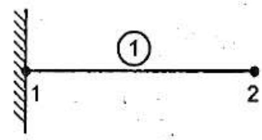
$$R_4 = -1.455\ kN$$

We know that, Reaction force is equivalent and opposite to the applied force.

Verification: $R_1 + R_2 + R_3 + R_4 = -6.546 + 0 + 0 - 1.455$

$= -8\ kN$ [Applied force]

Forces in each element:



For element 1: (Nodes 1, 2):

Finite element equation for element (1) is

$$\frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \frac{2 \times 10^{-4} \times 70 \times 10^{-6}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.935 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow 10^3\begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.935 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow 10^3(0 - 7 \times 0.935 \times 10^{-3}) = F_1$$

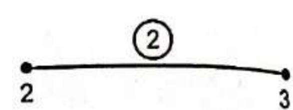
$$\Rightarrow 10^3(0 + 7 \times 0.935 \times 10^{-3}) = F_2$$

$$\Rightarrow F_1 = -6.546\ kN$$

$$F_2 = 6.546\ kN$$

For element (1): Force at node 1, $F_1 = -6.546\ kN$

Force at node 2, $F_2 = 6.546\ kN$



For element 2: (Nodes 2,3):

Finite element equation for element (2) is

$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \frac{2\times 10^{-4}\times 70\times 10^{-6}}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0.935\times 10^{-3}\\ 0.727\times 10^{-3}\end{Bmatrix}=\begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

$$\Rightarrow \qquad 10^3\begin{bmatrix}7 & -7\\ -7 & 7\end{bmatrix}\begin{Bmatrix}0.935\times 10^{-3}\\ 0.727\times 10^{-3}\end{Bmatrix}=\begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

$$\Rightarrow \quad 10^3(0-7\times 0.935\times 10^{-3}-7\times 0.727\times 10^{-3})=F_2$$

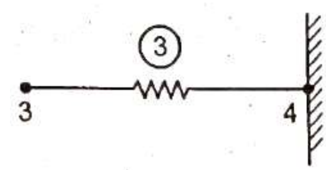
$$\Rightarrow \quad 10^3(0-7\times 0.935\times 10^{-3}+7\times 0.727\times 10^{-3})=F_3$$

$$\Rightarrow \qquad F_2=1.455\ kN$$

$$F_3=-1.455\ kN$$

For element (2): Force at node 2, $F_2=1.455\ kN$

Force at node 3, $F_3=-1.455\ kN$



For element 3: (Nodes 3, 4):

Finite element equation for spring element is given by,

$$k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$

$$2000\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0.727\times 10^{-3}\\ 0\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$

$$10^3\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}\begin{Bmatrix}0.727\times 10^{-3}\\ 0\end{Bmatrix}=\begin{Bmatrix}F_3\\ F_4\end{Bmatrix}$$

$$\Rightarrow \qquad 10^3(2\times 0.727\times 10^{-3}-0)=F_3$$

$$\Rightarrow \qquad 10^3(-2\times 0.727\times 10^{-3}+0)=F_4$$

$$\Rightarrow \qquad F_3=1.455\ kN$$

$$F_4=-1.455\ kN$$

For element (3): Force at node 3, $F_3=1.455\ kN$

Force at node 4, $F_4=-1.455\ kN$

Result: (i) Nodal displacements

$$u_1 = 0$$

$$u_2 = 0.935 \times 10^{-3} \text{ m}$$

$$u_3 = 0.727 \times 10^{-3} \text{ m}$$

$$u_4 = 0$$

(ii) Nodal forces in each element:

For element (1): F_1 = - 6.546 kN; F_2 = 6.548 kN

For element (2): F_2 = 1.455 kN

F_3 = - 1.455 kN

For element (3): F_3 = 1.455 kN

F_4 = - 1.455 kN

(iii) Nodal reactions in each element:

$$R_1 = -6546\ kN$$

$$R_2 = 0$$

$$R_3 = 0$$

$$R_4 = \ -1.455\ kN$$

2.11. TRUSSES

2.11.1. Introduction

A truss is defined as a structure, made up of several bars, riveted or welded together. The following assumptions are made while finding the forces in a truss.

(i) All the members are pin jointed.

(ii) The truss is loaded only at the joints.

(iii) The self-weight of the members are neglected unless stated.

A two dimensional truss element is shown in Fig.2.26

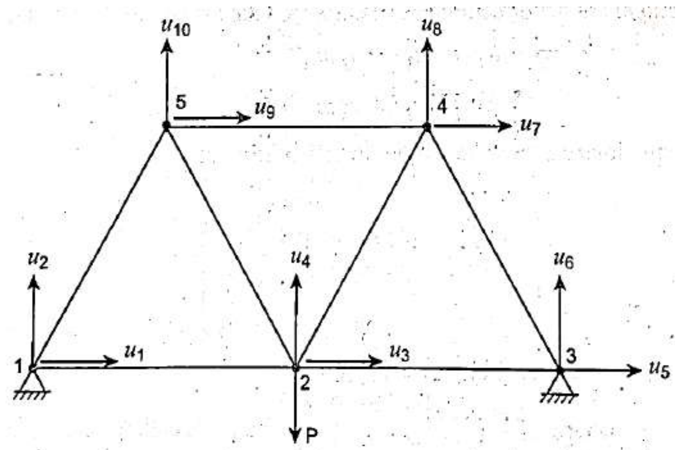


Fig. 2.26. A two dimensional truss element

2.11.2. Stiffness Matrix [K] for a Truss Element

Consider a two noded bar element as shown in Fig.2.27, for the analysis of trusses. This element is subjected to only axial forces. So, the displacements are only in the axial directions.

The nodal displacement for this bar element is given by,

$$\{u'\} = \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

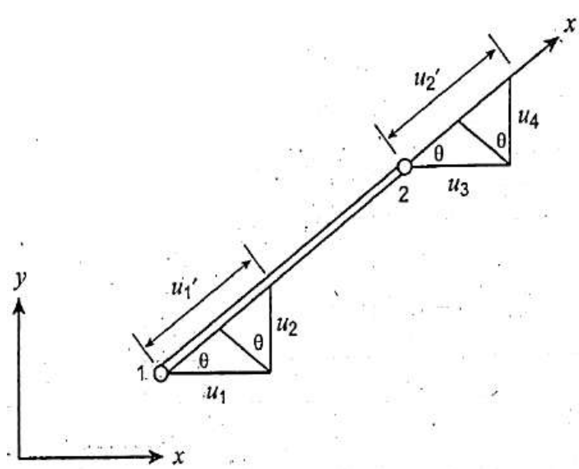


Fig. 2.27 Two noded bar element

From the Fig. 2.27, we know that,

$$u_1' = u_1 \cos\theta + u_2 \sin\theta$$

$$u_2' = u_3 \cos\theta + u_4 \sin\theta$$

Consider *l* and *m* are the direction consines, So, we take $l = \cos\theta$ and $m = \sin\theta$.

$$\Rightarrow \qquad u_1' = u_1 l + u_2 m \qquad \ldots(2.79)$$

$$u_2' = u_3 l + u_4 m \qquad \ldots(2.80)$$

The above equations can now be writer in matrix form as,

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \qquad \{u'\} = [L]\{u\} \qquad \ldots(2.81)$$

Where, $[\,L\,] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$ and it is called transformation matrix.

Referring to Fig. 2.28, let (x_1, y_1) and (x_2, y_2) be the co-ordinates of nodes 1 and 2 respectively. We can find *l*, *m* and l_e values by using the following formulae.

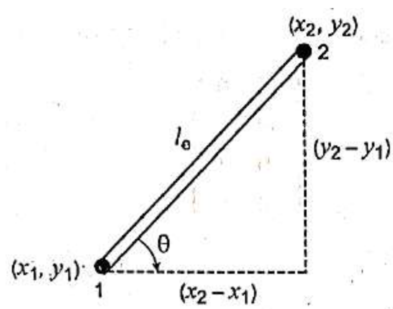


Fig. 2.28 Direction cosines

$$l = \cos\theta = \frac{x_2 - x_1}{l_e} \qquad \ldots(2.82)$$

$$m = \sin\theta = \frac{y_2 - y_1}{l_e} \qquad \ldots(2.83)$$

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \ldots(2.84)$$

From Fig. 2.28., we know that the truss is also a one dimensional two noded bar element. The stiffness matrix for two noded bar element is given by,

$$[K'] = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \dots (2.85)$$

We know that,

$$\text{Strain energy}, U = \frac{1}{2}\{u'\}^T [K'] \{u'\} \qquad \dots (2.86)$$

From equation (2.81), we know that,

$$\{u_1'\} = [\ L]\ \{u\}$$

Substitute $\{u_1'\}$ value in equation (2.86),

$$\Rightarrow U = \frac{1}{2}([L]\{u'\})^T [K'][L]\{u\}$$

$$= \frac{1}{2}[L]^T\ \{u\}^T\ [K']\ [L]\ \{u\}$$

$$U = \frac{1}{2}\ \{u\}^T\ \{u\}[K] \qquad \dots (2.87)$$

Where, $\quad [K] = \frac{1}{2}\ [L]^T\ [K']\ [L]$

Element stiffness matrix in global co-ordinates.

$$[K] = \frac{1}{2}\ [L]^T\ [K']\ [L] \qquad \dots (2.88)$$

Substitute [L] value from equation (2.81) and $[K']$ value from equation (2.85).

$$\Rightarrow [K] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$\because [L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix};\ [L]^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}$$

$$\Rightarrow [K] = \frac{A_e E_e}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} l-0 & m-0 & 0-l & 0-m \\ -l+0 & -m+0 & 0+l & 0+m \end{bmatrix}$$

$$[\because (2 \times \underbrace{2) \times (2} \times 4) = 2 \times 4]$$

$$= \frac{A_e E_e}{l_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}\begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e}\begin{bmatrix} l^2 & lm-0 & -l^2+0 & -lm+0 \\ lm-0 & m^2-0 & -ml+0 & -m^2+0 \\ 0-l^2 & 0-lm & 0+l^2 & 0+lm \\ 0-lm & 0-m^2 & 0+lm & 0+m^2 \end{bmatrix}$$

$$[\because (4 \times \underbrace{2) \times (2} \times 4) = 4 \times 4]$$

$$[K] = \frac{A_e E_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -ml & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \qquad \text{... (2.89)}$$

It may be noted that the stiffness matrix properties are satisfied.

1. [K] matrix is symmetric.
2. The sum of elements in any column is equal to zero.

2.11.3 Finite Element Equation for a Two Noded Truss element

We know that, finite element general equation is,

$$\{ F \} = [K] \{ u \} \qquad \text{…(2.90)}$$

Where, { F } is a element force vector [Column Matrix].

{ K } is a stiffness matrix [Row Matrix].

{ u } is a nodal displacements [Column Matrix].

We know that, For truss element, stiffness matrix,

$$[K] = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -ml & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Substitute [K] value in equation (2.90),

$$\Rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -ml & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad \text{... (2.91)}$$

This is a finite element equation for a truss element.

2.11.4 Solved Problems

EXAMPLE 2.20

For the two bar truss shown in fig.(i), determine the displacements of node 1 and the stress in element 1-3.

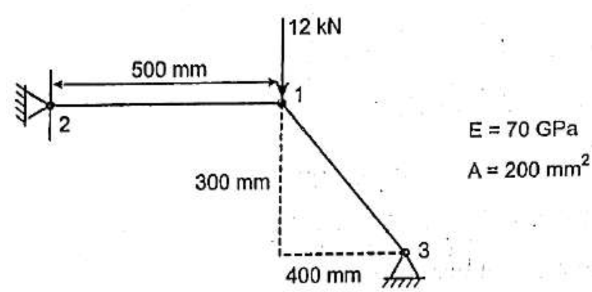


Fig. (i)

Given: Yound's modulus, E $= 70$ GPa $= 70 \times 10^9$ Pa

$= 70 \times 10^9$ N/m^2

$= 70 \times 10^9$ N/mm^2

Area, A $= 200$ mm^2

Point load at node 1 $= 12$ kN $= 12 \times 10^3$ N

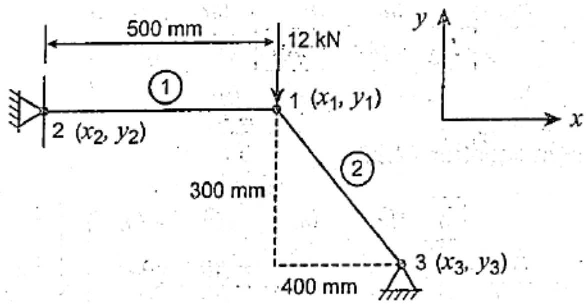


Fig. (ii)

To find: 1. Displacements of node, 1

2. stress in element, (2)

Solution: Consider node 1 as the origin.

The co-ordinates of various nodes are given below.

$$\text{Node } 1 = \overset{x_1}{(0,}\ \overset{y_1}{0)}$$

$$\text{Node } 2 = \overset{x_2}{(-500,}\ \overset{y_2}{0)}$$

$$\text{Node } 3 = \overset{x_3}{(400,}\ \overset{y_3}{-300)}$$

For element (1): Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(-500 - 0)^2 + (0 - 0)^2}$$

$$l_{e1} = 500\ mm$$

Direction consines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}}$

$$= \frac{-500 - 0}{500}$$

$$l_1 = -1$$

$$m_1 = \frac{y_2 - y_1}{l_{e1}}$$

$$= \frac{0 - 0}{500}$$

$$m_1 = 0$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(400 - 0)^2 + (-300 - 0)^2}$$

$$l_{e2} = 500\ mm$$

Direction consines, $l_2 = \dfrac{x_3 - x_1}{l_{e2}}$

$$= \frac{400}{500}$$

$$l_2 = 0.8$$

$$m_l = \frac{y_3 - y_1}{l_{e2}}$$

$$= \frac{-300 - 0}{500}$$

$$m_l = -0.6$$

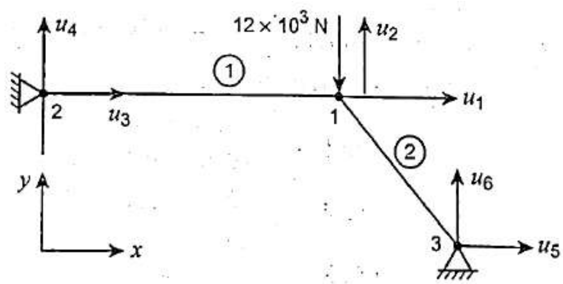


Fig. (iii)

For element (1): Displacements $u_1, u_2, u_3 and\ u_4$

Stiffness matrix [K] for a truss element is given by,

$$[K]_1 = \frac{A_1 E_1}{l_{e1}} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

[From equation no. (2.89)]

$$= \frac{200 \times 70 \times 10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Global numbers

$$[K]_1 = 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \qquad \ldots(1)$$

For element (2): Displacements $u_1, u_2, u_5 \, and \, u_6$

Stiffness matrix [K]$_2$ $= \dfrac{A_2 E_2}{l_{e2}} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_1^2 & -l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_1 m_1 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & m_2^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$

$$= \frac{200 \times 70 \times 10^3}{500} \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$[K]_2 = 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \qquad \ldots(2)$$

Assemble the stiffness matrix [K], i.e., assemble the equation (1) and (2)

[K] = 28 × 10^3

1	2	3	4	5	6	
1 + 0.64	1 + -0.48	0	0	0.64	0.48	1
1 + -0.48	1 + 0.36	0	0	0.48	-0.36	2

-1	0	1	0	0	0	3
0	0	0	0	0	0	4
-0.64	0.48	0	0	0.64	-0.48	5
0.48	-0.36	0	0	-0.48	0.36	6

$$\Rightarrow [K] = 28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \quad ...(3)$$

[Note: The given truss element has 3 nodes and each node has 2 degree of freedom. So, total degrees of freedom are 6 ($u_1, u_2, u_3, u_4, u_5 and\ u_6$). Hence the stiffness matrix size is 6 × 6.]

We know that, General finite element equation is

$$\{ F \} = [K] \{ u \}$$

$$\Rightarrow [K] \{ u \} = \{ F \}$$

$$28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \quad ...(4)$$

Applying boundary conditions [Refer Fig. (iii)]:

(i) Node 2 is fixed. So, $u_3 = u_4 = 0$

(ii) Node 3 is fixed. So, $u_5 = u_6 = 0$

(iii) A point load of 12×10^3 N is acting at node 1 in downward direction.

So, $F_2 = - 12 \times 10^3$ N

(iv) Self-weight is neglected. So, $F_1 = F_3 = F_4 = F_5 = F_6 = 0$

Substitute the above values in equation (4)

$$\Rightarrow 28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -12 \times 10^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation $u_3 = u_4 = u_5 = u_6 = 0$. So, delete third row third column, fourth row, fourth column, fifth row fifth column and sixth row sixth column of [K] matrix. Hence the equation reduces to

$$\Rightarrow \quad 28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -12 \times 10^3 \end{Bmatrix}$$

$$\Rightarrow \quad 28 \times 10^3 (1.64 u_1 - 0.48\, u_2) = 0 \qquad \text{... (5)}$$

$$\Rightarrow \quad 28 \times 10^3 (-0.48\, u_1 - 0.36\, u_2) = -12 \times 10^3 \qquad \text{... (6)}$$

Equation (6) × 3.4166 ⇒

$$28 \times 10^3 (-1.64\, u_1 - 1.23\, u_2) = -41 \times 10^3 \qquad \text{... (7)}$$

$$28 \times 10^3 (1.64\, u_1 - 0.48\, u_2) = 0 \qquad \text{... (5)}$$

Solving, $\quad 28 \times 10^3 (0.75\, u_2) = -41 \times 10^3$

$$\Rightarrow u_2 = -1.952\, mm$$

Substitute u_2 value in equation (5),

$$28 \times 10^3 (1.64 u_1 - 0.48\,(-1.952)) = 0$$

$$\Rightarrow \quad 1.64\, u_1 - 0.48\,(-1.952) = 0$$

$$\Rightarrow u_1 = -0.571\, mm$$

For element (1):

$$Stress, \sigma = \frac{E}{l_e} [-l \quad -m \quad l \quad m] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

For element (1): Displacements $u_1, u_2, u_5 and\ u_6$

$$Stress, \sigma_2 = \frac{E}{l_e}[-l_2 \quad -m_2 \quad l_2 \quad m_2] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{70 \times 10^3}{500}[-0.8 \quad -0.6 \quad 0.8 \quad -0.6] \begin{Bmatrix} -0.571 \\ -1.952 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 140\ [(-0.8) \times (-0.571) + 0.6 \times (-1.952) + 0 + 0]$$

$$\sigma_2 = -100\ N/mm^2 \quad \text{[compressive stress]}$$

Result: 1. Displacement of node 1

$$u_1 = -0.571\ mm$$

$$u_2 = -1.952\ mm$$

2. Stress in element (2)

$$\sigma_2 = -100\ N/mm^2 \quad \text{[compressive stress]}$$

Example 2.21

Consider a three bar truss as shown in Fig.(i). It is given that E = 2 × 10^5 N/mm^2. Calculate the following:

(i) Nodal displacements.

(ii) Stress in each member.

(iii) Reactions at the support.

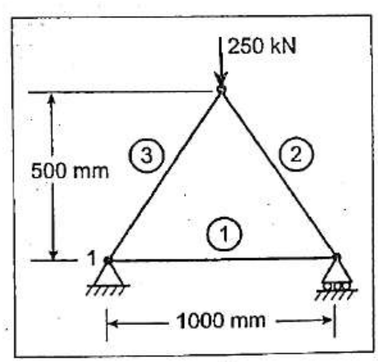


Fig. (i)

Take Area of element (1) = 2000mm^2

Area of element (2) = 2500 mm^2

Area of element (3) = 2500 mm^2

Given: Young's modulus, E = 2 X10^5 N/mm^2

Area of element (1) = 2000 mm^2

Area of element (2) = 2500 mm^2

Area of element (3) = 2500 mm^2

Point load, P = - 250 kN = - 250 × 10^3 N

= - 2.5 × 10^5 N [∵ Load is downward direction]

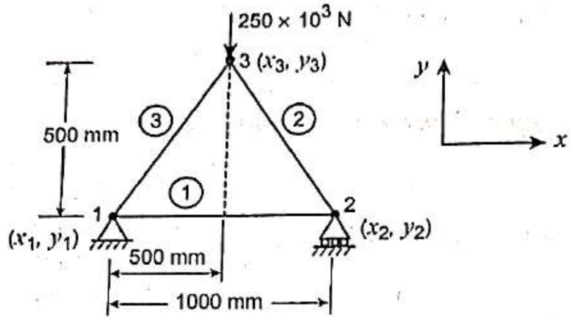


Fig. (ii)

[Node: Number with circle = Element; Number without circle = Node]

To find:

(i) Nodal displacements, $u_1, u_2, u_3, u_4, u_5 and\ u_6$.

(ii) Stress in each member, $\sigma_1, \sigma_2\ and\ \sigma_3$

(iii) Reactions at the support $R_1, R_2, R_3, R_4, R_5 and\ R_6$.

Solution: Consider node 1 as the origin.

The co-ordinates of various nodes are given below:

$$\text{Node } 1 = \begin{matrix} x_1 & y_1 \\ (0, & 0) \end{matrix}$$

$$\text{Node } 2 = \begin{matrix} x_2 & y_2 \\ (1000, & 0) \end{matrix}$$

$$\text{Node } 3 = \begin{matrix} x_3 & y_3 \\ (500, & 500) \end{matrix}$$

For element (1): Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(1000 - 0)^2 + (0 - 0)^2}$$

$$l_{e1} = 1000\ mm$$

Direction consines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}}$

$$= \frac{1000 - 0}{1000}$$

$$l_1 = 1$$

$$m_1 = \frac{y_2 - y_1}{l_{e1}}$$

$$= \frac{0 - 0}{1000}$$

$$m_1 = 0$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(500 - 1000)^2 + (500 - 0)^2}$$

$$l_{e2} = 707.107\ mm$$

Direction consines, $l_2 = \dfrac{x_3 - x_1}{l_{e2}}$

$$= \frac{500 - 1000}{707.107}$$

$$l_2 = -0.707$$

$$m_2 = \frac{y_3 - y_1}{l_{e2}}$$

$$= \frac{500 - 0}{707.107}$$

$$m_2 = 0.707$$

For element (3): Length, $l_{e3} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(500 - 0)^2 + (500 - 0)^2}$$

$$l_{e3} = 707.107\ mm$$

Direction consines, $l_3 = \dfrac{x_3 - x_1}{l_{e2}}$

$$= \frac{500 - 0}{707.107}$$

$$l_3 = 0.707$$

$$m_3 = \frac{y_3 - y_1}{l_{e2}}$$

$$= \frac{500 - 0}{707.107}$$

$$m_3 = 0.707$$

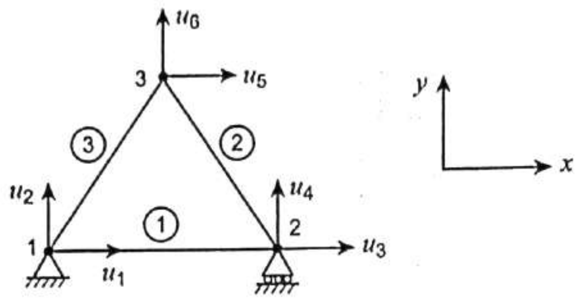


Fig. (iii)

For element (1): Displacements $u_1, u_2, u_3 and\ u_4$ [Refer Fig. (iii)]

Stiffness matrix [K] for a truss element is given by,

$$[K]_1 = \frac{A_1E_1}{l_{e1}}\begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix}$$

$$= \frac{2000 \times 2 \times 10^5}{1000}\begin{bmatrix} (1)^2 & 1 \times 0 & -(1)^2 & -(1 \times 0) \\ 1 \times 0 & (0)^2 & -(1 \times 0) & -(0)^2 \\ -(1)^2 & -(1 \times 0) & (1)^2 & 1 \times 0 \\ 0 & -(0)^2 & 1 \times 0 & (0)^2 \end{bmatrix}$$

$$= 4 \times 10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_1 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \qquad \ldots(1)$$

For element (2): Displacements $_{1}, \boldsymbol{u_2}, \boldsymbol{u_5} \boldsymbol{and}\ \boldsymbol{u_6}$:

$\boldsymbol{u}$

$$\text{Stiffness matrix } [\text{ K }]_2 = \frac{A_2E_2}{l_{e2}}\begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_1^2 & -l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_1m_1 \\ -l_2m_2 & -m_2^2 & l_2m_2 & m_2^2 \end{bmatrix}$$

$$= \frac{2500 \times 2 \times 10^5}{707.107}\begin{bmatrix} (-0.707)^2 & -0.707 \times 0.707 & (-0.707)^2 & 0.707 \times 0.707 \\ -0.707 \times 0.707 & (0.707)^2 & 0.707 \times 0.707 & (-0.707)^2 \\ -(-0.707)^2 & -0.707 \times 0.707 & (-0.707)^2 & -0.707 \times 0.707 \\ 0.707 \times 0.707 & -(0.707)^2 & -0.707 \times 0.707 & (0.707)^2 \end{bmatrix}$$

$$= 28 \times 10^3 \begin{bmatrix} 0.4998 & -0.4998 & -0.4998 & 0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ -0.4998 & 0.4998 & 0.4998 & -0.4998 \\ 0.4998 & -0.4998 & -0.4998 & 0.4998 \end{bmatrix}$$

$$[K]_2 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} 3.534 & -3.534 & -3.534 & 3.534 \\ -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & 3.534 & 3.534 & -3.534 \\ 3.534 & -3.534 & -3.534 & 3.534 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{array} \quad \ldots(2)$$

For element (3): Displacements $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_5} \boldsymbol{and}\ \boldsymbol{u_6}$:

$$\text{Stiffness matrix } [\text{ K }]_3 = \frac{A_3E_3}{l_{e3}}\begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & -l_3m_3 & -m_2^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}$$

$$= \frac{2500 \times 2 \times 10^5}{707.107}\begin{bmatrix} (-0.707)^2 & 0.707 \times 0.707 & -(0.707)^2 & -0.707 \times 0.707 \\ -0.707 \times 0.707 & (0.707)^2 & 0.707 \times 0.707 & -(0.707)^2 \\ -(0.707)^2 & -0.707 \times 0.707 & (0.707)^2 & 0.707 \times 0.707 \\ -0.707 \times 0.707 & -(0.707)^2 & 0.707 \times 0.707 & (0.707)^2 \end{bmatrix}$$

$$= 28 \times 10^3 \begin{bmatrix} 0.4998 & 0.4998 & -0.4998 & -0.4998 \\ 0.4998 & 0.4998 & -0.4998 & -0.4998 \\ -0.4998 & -0.4998 & 0.4998 & 0.4998 \\ -0.4998 & -0.4998 & 0.4998 & 0.4998 \end{bmatrix}$$

$$[K]_3 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 3.534 & -3.534 & -3.534 & 3.534 \\ -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & 3.534 & 3.534 & -3.534 \\ 3.534 & -3.534 & -3.534 & 3.534 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \quad \ldots(3)$$

Assemble the stiffness matrix [K], i.e., assemble the equation (2) and (3)

$[K] = 1 \times 10^5$

1	2	3	4	5	6	
4 + 3.534	0 + 3.534	-4	0	-3.534	-3.534	1
0 + 3.534	0 + 3.534	0	0	-3.534	-3.534	2
-4	0	4 + 3.534	0 - 3.534	-3.534	3.534	3
0	0	0 - 3.534	0 + 3.534	3.534	-3.534	4
-3.534	-3.534	-3.534	-3.534	-3.534 + 3.534	-3.534 + 3.534	5
-3.534	-3.534	-3.534	-3.534	-3.534 + 3.534	-3.534 + 3.534	6

$$\Rightarrow [K] = 1\text{x}10^5 \begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ -0.48 & 0.36 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068 \end{bmatrix} \ldots (4)$$

[Note: The given truss element has 3 nodes and each node has 2 degree of freedom. So, total degrees of freedom are 6 ($u_1, u_2, u_3, u_4, u_5 and\ u_6$). Hence the stiffness matrix size is 6 × 6.]

We know that, General finite element equation is

$$\{ F \} = [K] \{ u \}$$

$$\Rightarrow [K] \{ u \} = \{ F \}$$

$$1 \times 10^5 \begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ -0.48 & 0.36 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

... (5)

Applying boundary conditions [Refer Fig. (iii)]:

i) Node 1 is fixed. So, $u_1 = u_2 = 0$

ii) Node 2 is moving in *x* direction. So, $u_3 \neq 0\ and\ u_4 = 0$

iii) At node 3, Point load of 250 × 10^3 N is acting in downward direction.
So, F_6 = - 250 × 10^3 N = -2.5 × 10^5 N

iv) Self-weight is neglected. So, $F_1 = F_2 = F_3 = F_4 = F_5 = 0$.

Substitute u_1, u_2, u_4, F_1, F_2, F_3, F_4, F_5 and F_6 values in equation (5).

$$1 \times 10^5 \begin{bmatrix} 7.534 & 3.534 & -4 & 0 & -3.534 & -3.534 \\ -0.48 & 0.36 & 0 & 0 & -3.534 & -3.534 \\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534 \\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534 \\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0 \\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ 0 \\ u_5 \\ u_6 \end{Bmatrix}$$
$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2.5 \times 10^5 \end{Bmatrix}$$

In the above equation $u_1 = 0$. So, delete first row first column of [K] matrix. $u_2 = 0\ and\ u_4 = 0$. Hence delete second row, second column, fourth row and fourth column of [K] matrix.

The final reduced equation is,

$$1 \times 10^5 \begin{bmatrix} 7.534 & -3.534 & 3.534 \\ -3.534 & 7.068 & 0 \\ 3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2.5 \times 10^5 \end{Bmatrix}$$

$$\begin{bmatrix} 7.534 & -3.534 & 3.534 \\ -3.534 & 7.068 & 0 \\ 3.534 & 0 & 7.068 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2.5 \end{Bmatrix} \quad \text{... (6)}$$

By using Gaussian elimination method, we can find u_3, u_5 *and* u_6 values.

$$Let, \quad \left(\begin{array}{ccc|c} 7.534 & -3.534 & 3.534 & 0 \\ -3.534 & 7.068 & 0 & 0 \\ 3.534 & 0 & 7.068 & -2.5 \end{array}\right)$$

$$\Rightarrow \quad \left(\begin{array}{ccc|c} 7.534 & -3.534 & 3.534 & 0 \\ -3.534 & 7.068 & 0 & 0 \\ 3.534 & 0 & 7.068 & -2.5 \end{array}\right) R_1 \to \frac{R_1}{7.534}$$

$$\Rightarrow \quad \left(\begin{array}{ccc|c} 1 & -0.469 & 0.469 & 0 \\ 0 & 5.410 & 1.657 & 0 \\ 3.534 & 0 & 7.068 & -2.5 \end{array}\right) R_1 \to R_2 + 3.534 R_1$$

$$\Rightarrow \quad \begin{pmatrix} 1 & -0.469 & 0.469 & 0 \\ 0 & 5.410 & 1.657 & 0 \\ 0 & 0 & 4.902 & -2.5 \end{pmatrix} R_3 \to R_3 - \frac{R_2}{3.2649}$$

Now, we obtained the required triangular matrix

$$\Rightarrow \quad \begin{bmatrix} 1 & -0.469 & 0.469 \\ 0 & 5.410 & 1.657 \\ 0 & 0 & 4.902 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2.5 \end{Bmatrix}$$

$$\Rightarrow \quad 4.902\ u_6 = -2.5$$

$$u_6 = -0.5099\ mm$$

$$\Rightarrow \quad 5.410\, u_5 + 1.657\, u_6 = 0$$

$$5.410\, u_5 + 1.657\,(-0.5099) = 0$$

$$u_5 = 0.1562\ mm$$

$$\Rightarrow \qquad u_3 - 0.469\,u_5 + 0.469\,u_6 = 0$$

$$u_3 - 0.469\,(0.1562) + 0.469(-0.5099) = 0$$

$$u_3 - 0.3124 = 0$$

$$u_3 = 0.3124\,mm$$

We know that,

$$Stress, \sigma = \frac{E}{l_e}[-l \quad -m \quad l \quad m]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

For element (1):

$$Stress, \sigma_1 = \frac{E}{l_e}[-l_1 \quad -m_1 \quad l_1 \quad m_1]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{2 \times 10^5}{1000}[-1 \quad -0 \quad 1 \quad 0]\begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \end{Bmatrix}$$

$$= \frac{2 \times 10^5}{1000}\ [0 - 0 + 0.3124 + 0]$$

$$= \frac{2 \times 10^5}{1000} \times 0.3124$$

$$\sigma_2 = 62.48\,N/mm^2 \quad \text{[Tensile stress]}$$

For element (2):

$$Stress, \sigma_2 = \frac{E_2}{l_{e2}}[-l_2 \quad -m_2 \quad l_2 \quad m_2]\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= \frac{2 \times 10^5}{707.107}[0.707 \quad -0.707 \quad -0.707 \quad 0.707]\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= 282.842\begin{bmatrix}0.707 & -0.707 & -0.707 & 0.707\end{bmatrix}\begin{Bmatrix}0.3124\\0\\0.1562\\-0.5099\end{Bmatrix}$$

$$= 282.842[0.707 \times 0.3124 - 0.707 \times 0 - 0.707 \times 0.1562 + 0.707\,](-0.5099)$$

$$= 282.842(-0.250062)$$

$$\sigma_2 = -70.729\ N/mm^2 \quad \text{[Compressive stress]}$$

For element (3):

$$Stress, \sigma_3 = \frac{E_3}{l_{e3}}\begin{bmatrix}-l_3 & -m_3 & l_3 & m_3\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_5\\u_6\end{Bmatrix}$$

$$= \frac{2 \times 10^5}{707.107}\begin{bmatrix}-0.707 & -0.707 & 0.707 & 0.707\end{bmatrix}\begin{Bmatrix}0\\0\\0.1562\\-0.5099\end{Bmatrix}$$

$$= 282.842[0 - 0 + 0.707(0.1562) + 0.707\,](-0.5099)$$

$$= 282.842(-0.250065)$$

$$\sigma_3 = -70.729\ N/mm^2 \quad \text{[Compressive stress]}$$

Reaction forces: we know that

Reaction force, { R } = [K] {u*} – { F }

$$\begin{Bmatrix}R_1\\R_2\\R_3\\R_4\\R_5\\R_6\end{Bmatrix} = 1 \times 10^5\begin{bmatrix}7.534 & 3.534 & -4 & 0 & -3.534 & -3.534\\ -0.48 & 0.36 & 0 & 0 & -3.534 & -3.534\\ -4 & 0 & 7.534 & -3.534 & -3.534 & 3.534\\ 0 & 0 & -3.534 & 3.534 & 3.534 & -3.534\\ -3.534 & -3.534 & -3.534 & 3.534 & 7.068 & 0\\ -3.534 & -3.534 & 3.534 & -3.534 & 0 & 7.068\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{Bmatrix} - \begin{Bmatrix}F_1\\F_2\\F_3\\F_4\\F_5\\F_6\end{Bmatrix}$$

$$\Rightarrow \{R_1\} = 1 \times 10^5 [7.534 \quad 3.534 \quad -4 \quad 0 \quad -3.534 \quad -3.534\,] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \{F_1\}$$

$$= 1 \times 10^5 [7.534 \quad 3.534 \quad -4 \quad 0 \quad -3.534 \quad -3.534\,] \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix} - \{0\}$$

$$= 1 \times 10^5 [0 + 0 - 4 \times 0.3124 + 0 - 3.534 \times 0.1562 + (-3.534) \times (-0.5099)\,]$$

$$\{R_1\} = 0$$

$$\Rightarrow \{R_2\} = 1 \times 10^5 [3.534 \quad 3.534 \quad 0 \quad 0 \quad -3.534 \quad -3.534\,] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \{F_2\}$$

$$= 1 \times 10^5 [3.534 \quad 3.534 \quad 0 \quad 0 \quad -3.534 \quad -3.534\,] \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix} - \{0\}$$

$$= 1 \times 10^5 [0 + 0 + 0 - 3.534 \times 0.1562 + (-3.534) \times (-0.5099)\,]$$

$$\{R_2\} = 1.249 \times 10^5 N$$

$$\Rightarrow \{R_3\} = 1 \times 10^5 [-4 \quad 0 \quad 7.534 \quad -3.534 \quad -3.534 \quad 3.534] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \{F_3\}$$

$$= 1 \times 10^5 [-4 \quad 0 \quad 7.534 \quad -3.534 \quad -3.534 \quad 3.534\,] \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix} - \{0\}$$

$$= 1 \times 10^5[0 + 0 + 7.534 \times 3.534 + 0 + (-3.534) \times 0.1562 + 3.534 \times (-0.5099)\ ]$$

$$\{R_3\} = 0$$

$$\Rightarrow \{R_4\} = 1 \times 10^5[0 \quad 0 \quad -3.534 \quad 3.534 \quad -3.534 \quad -3.534] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \{F_4\}$$

$$= 1 \times 10^5[0 \quad 0 \quad -3.534 \quad 3.534 \quad 3.534 \quad -3.534\ ] \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix} - \{0\}$$

$$= 1 \times 10^5[0 + 0 - 3.534 \times 0.3124 + 0 + 3.534 \times 0.1562 + (3.534) \times (-0.5099)\ ]$$

$$\{R_4\} = 1.249\ \times 10^5 N$$

$$\Rightarrow \{R_5\} = 1 \times 10^5[-3.534 \quad -3.534 \quad -3.534 \quad 3.534 \quad 7.068 \quad 0] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \{F_5\}$$

$$= 1 \times 10^5[-3.534 \quad -3.534 \quad -3.534 \quad 3.534 \quad 7.068 \quad 0\ ] \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix} - \{0\}$$

$$= 1 \times 10^5[0 + 0 + (-3.534) \times 0.3124 + 0 + 7.068 \times 0.1562 + 0\ ]$$

$$\{R_5\} = 0$$

$$\Rightarrow \{R_6\} = 1 \times 10^5[-3.534 \quad -3.534 \quad 3.534 \quad -3.534 \quad 0 \quad 7.068] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \{F_6\}$$

$$= 1 \times 10^5[-3.534 \quad -3.534 \quad 3.534 \quad -3.534 \quad 0 \quad 7.068] \begin{Bmatrix} 0 \\ 0 \\ 0.3124 \\ 0 \\ 0.1562 \\ -0.5099 \end{Bmatrix}$$

$$- 1 \times 10^5\{-2.5\}$$

$$= 1 \times 10^5[0 + 0 + 3.534 \times 0.3124 + 0 + 0 + 7.068 \times (-0.5099)]$$
$$- 1 \times 10^5\{-2.5\}$$

$$= -2.5 \times 10^5 + 2.5 \times 10^5$$

$$\{R_6\} = 0$$

We know that, Reaction force is equivalent and opposite to the applied force.

Verification: $R_1+R_2+R_3+R_4+R_5+R_6 = 0 + 1.249 \times 10^5 + 0 + 1.249 \times 10^5+0+0$

$= 2.5 \times 10^5$N [Applied force]

Result: 1. Nodal displacement

$$u_1 = 0$$

$$u_2 = 0$$

$$u_3 = 0.3124$$

$$u_4 = 0$$

$$u_5 = 0.1562$$

$$u_6 = -0.5099$$

2. Stress in each member:

$$\sigma_1 = 62.48\ N/mm^2 \quad \text{[Tensile]}$$

$$\sigma_2 = -70.729\ N/mm^2 \quad \text{[compressive]}$$

$$\sigma_3 = -70.729\ N/mm^2 \quad \text{[compressive]}$$

3. Reaction forces:

$$R_1 = 0$$

$$R_2 = 1.249 \times 10^5 N$$

$$R_3 = 0$$

$$R_4 = 1.249 \times 10^5 N$$

$$R_5 = 0$$

$$R_6 = 0$$

Example 2.22

Consider a four bar truss as shown in Fig. (i). It is given that E = 2 x10^5 N/mm^2 and A_e = 625 mm^2 for all elements.

(i) Determine the element stiffness matrix for each element.

(ii) Assemble the structural stiffness matrix [K] for the entire truss.

(iii) Solve for the nodal displacement.

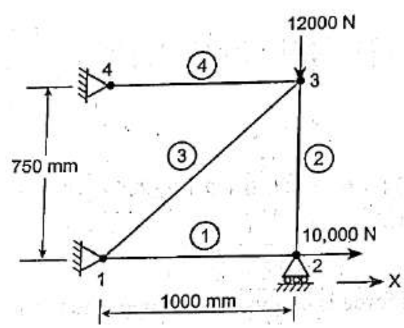


Fig. (i)

Given: Young's Modulus, E = 2 X10^5 N/mm^2

Area of each element (1), A_e = 625mm^2

Load acting at node 3 = - 12000 N [∵ Load is downward direction]

Load acting at node 2 = 10000 N

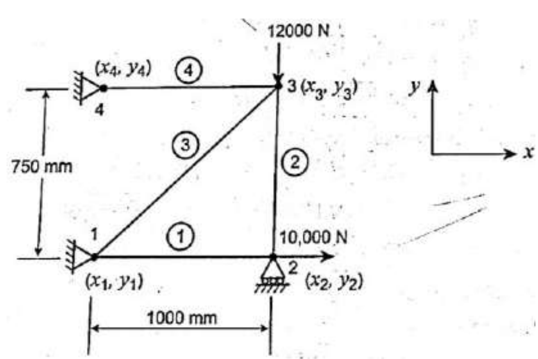


Fig.(ii)

[Node: Number with circle denotes element & Number without circle denotes Node]

To find:

1. Element stiffness matrix for each element
2. Global stiffness matrix [K]
3. Nodal displacements $u_1, u_2, u_3, u_4, u_5, u_6, u_7 \; and \; u_8$

Solution: Consider node 1 as the origin.

The co-ordinated of various nodes are given below:

$$\text{Node } 1 = (\overset{x_1}{0}, \; \overset{y_1}{0})$$

$$\text{Node } 2 = (\overset{x_2}{1000}, \; \overset{y_2}{0})$$

$$\text{Node } 3 = (\overset{x_3}{1000}, \; \overset{y_3}{750})$$

$$\text{Node } 4 = (\overset{x_4}{0}, \; \overset{y_4}{750})$$

For element (1): Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [From equation No. (2.84)]

$$= \sqrt{(1000 - 0)^2 + (0 - 0)^2}$$

$$l_{e1} = 1000 \, mm$$

Direction consines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}}$ [From equation No. (2.82)]

$$= \frac{1000 - 0}{1000}$$

$$l_1 = 1$$

$m_1 = \dfrac{y_2 - y_1}{l_{e1}}$ [From equation No. (2.83)]

$$= \frac{0 - 0}{1000}$$

$$m_1 = 0$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

$$= \sqrt{(1000 - 1000)^2 + (750 - 0)^2}$$

$$l_{e2} = 750\ mm$$

Direction consines, $l_2 = \dfrac{x_3 - x_2}{l_{e2}}$

$$= \frac{1000 - 1000}{750}$$

$$l_2 = 0$$

$$m_2 = \frac{y_3 - y_2}{l_{e2}}$$

$$= \frac{750 - 0}{750}$$

$$m_2 = 1$$

For element (3): Length, $l_{e3} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(1000 - 0)^2 + (750 - 0)^2}$$

$$l_{e3} = 1250\ mm$$

Direction consines, $l_3 = \dfrac{x_3 - x_1}{l_{e2}}$

$$= \frac{1000 - 0}{1250}$$

$$l_3 = 0.8$$

$$m_3 = \frac{y_3 - y_1}{l_{e2}}$$

$$= \frac{750 - 0}{1250}$$

$$m_3 = 0.6$$

For element (4): Length, $l_{e4} = \sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2}$

$$= \sqrt{(1000 - 0)^2 + (750 - 750)^2}$$

$$l_{e4} = 1000\ mm$$

Direction consines, $l_4 = \dfrac{x_3 - x_4}{l_{e2}}$

$$= \frac{1000 - 0}{1000}$$

$$l_4 = 1$$

$$m_3 = \frac{y_3 - y_4}{l_{e2}}$$

$$= \frac{750 - 750}{1000}$$

$$m_4 = 0$$

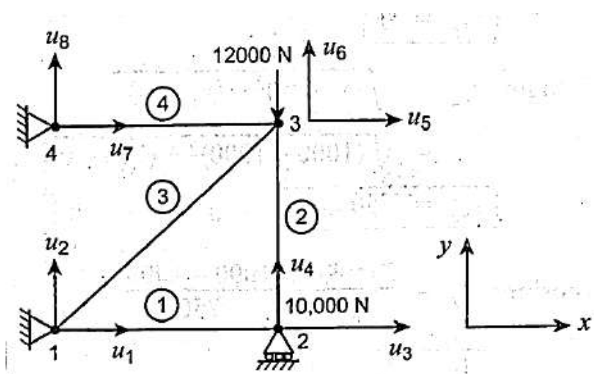


Fig. (iii)

For element (1): Displacements $u_1, u_2, u_3 and\ u_4$

Stiffness matrix [K] for a truss element is given by,

$$[K]_1 = \frac{A_1E_1}{l_{e1}}\begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix}$$

$$= \frac{625 \times 2 \times 10^5}{1000}\begin{bmatrix} (1)^2 & 1 \times 0 & -(1)^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(1)^2 & 0 & (1)^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 1.25 \times 10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_1 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \qquad \ldots(1)$$

For element (2): Displacements $u_3, u_4, u_5 and\ u_6$:

$$\text{Stiffness matrix } [\text{ K }]_2 = \frac{A_2E_2}{l_{e2}} \begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_1^2 & -l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_1m_1 \\ -l_2m_2 & -m_2^2 & l_2m_2 & m_2^2 \end{bmatrix}$$

$$= \frac{625 \times 2 \times 10^5}{750} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= 1.666 \times 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K]_2 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.666 & 0 & -1.666 \\ 0 & 0 & 0 & 0 \\ 0 & -1.666 & 0 & 1.666 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{array} \qquad \ldots(2)$$

For element (3): Displacements $u_1, u_2, u_5 and\ u_6$:

$$\text{Stiffness matrix } [\text{ K }]_3 = \frac{A_3E_3}{l_{e3}} \begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & -l_3m_3 & -m_2^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}$$

$$= \frac{625 \times 2 \times 10^5}{750} \begin{bmatrix} (0.8)^2 & 0.8 \times 0.6 & -(0.8)^2 & -0.8 \times 0.6 \\ 0.8 \times 0.6 & (0.6)^2 & -0.8 \times 0.6 & -(0.6)^2 \\ -(0.8)^2 & -0.8 \times 0.6 & (0.8)^2 & 0.8 \times 0.6 \\ -0.8 \times 0.6 & -(0.6)^2 & 0.8 \times 0.6 & (0.6)^2 \end{bmatrix}$$

$$[K]_3 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & 0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{6} \end{array} \quad \ldots(3)$$

For element (3): Displacements $u_7, u_8, u_5 \, and \, u_6$:

$$\text{Stiffness matrix } [\text{ K }]_4 = \frac{A_4 E_4}{l_{e4}} \begin{bmatrix} l_4^2 & l_4 m_4 & -l_4^2 & -l_4 m_4 \\ l_4 m_4 & m_4^2 & -l_4 m_4 & -m_4^2 \\ -l_4^2 & -l_4 m_4 & l_4^2 & l_4 m_4 \\ -l_4 m_4 & -m_4^2 & l_4 m_4 & m_4^2 \end{bmatrix}$$

$$= \frac{625 \times 2 \times 10^5}{750} \begin{bmatrix} (1)^2 & 0 & -(1)^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(1)^2 & 0 & 1^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 1.25 \times 10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_4 = 1 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{7} & \mathbf{8} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{7} \\ \mathbf{8} \\ \mathbf{5} \\ \mathbf{6} \end{array} \quad \ldots(4)$$

Assemble the stiffness matrix [K], i.e., assemble the equation (1), (2), (3) and (4)

$[\,K\,] = 1 \times 10^5$

1	2	3	4	5	6	7	8	
1.25 + 0.64	0 + 0.48	-1.25	0	-0.64	-0.48	0	0	1
0 + 0.48	0 + 0.36	0	0	-0.48	-0.36	0	0	2
-1.25	0	1.25 + 0	0 + 0	0	0	0	0	3
0	0	0 + 0	0 + 1.666	0	- 1.666	0	0	4
-0.64	-0.48	0	0	0 + 0.64 + 1.25	0 + 0.48 + 0	-1.25	0	5
-0.48	-0.36	0	- 1.666	0 + 0.48 + 0	1.666 + 0.36 + 0	0	0	6
0	0	0	0	-1.25	0	1.25	0	7
0	0	0	0	0	0	0	0	8

$$\Rightarrow [K] = 1 \times 10^5 \begin{bmatrix} 1.89 & 0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0 \\ 048 & 0.36 & 0 & 0 & -0.48 & -0.36 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.666 & 0 & -1.666 & 0 & 0 \\ -0.64 & -0.48 & 0 & 0 & 1.89 & 0.48 & -1.25 & 0 \\ -0.48 & -0.36 & 0 & -1.666 & 0.48 & 2.026 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

... (5)

We know that, General finite element equation is

$$\{ F \} = [K] \{ u \}$$

$$\Rightarrow [K] \{ u \} = \{ F \}$$

$$\Rightarrow 1 \times 10^5 \begin{bmatrix} 1.89 & 0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0 \\ 048 & 0.36 & 0 & 0 & -0.48 & -0.36 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.666 & 0 & -1.666 & 0 & 0 \\ -0.64 & -0.48 & 0 & 0 & 1.89 & 0.48 & -1.25 & 0 \\ -0.48 & -0.36 & 0 & -1.666 & 0.48 & 2.026 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} \qquad \ldots (6)$$

Applying boundary conditions [Refer Fig. (iii)]:

1. Node 1 is fixed. So, $u_1 = u_2 = 0$.
2. Node 4 is fixed. So, $u_7 = u_8 = 0$.
3. Node 2 is moving in *x* direction. So, $u \neq 0$ *and* $u_4 = 0$.
4. At node 3, point load of 12000 N is acting in downward direction. So, F_6 = 12000 N.
5. At node 2, point load of 10000 N is acting in x direction. So, F_3 = 10000N.
6. Self-weight is neglected. So, $F_1 = F_2 = F_4 = F_5 = F_7 = F_8 = 0$.

Substitute boundary conditions values in equation (6). $u_1 = u_2 = u_4 = u_7 = u_8 = 0$. So, delete first row first column, second row second column, fourth row fourth column, seventh row seventh column and eighth row eighth column of [K] matrix.

The final reduced equation is,

$$1 \times 10^5 \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1.89 & 0.48 \\ 0 & 0.48 & 2.026 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 10{,}000 \\ 0 \\ -12{,}000 \end{Bmatrix}$$

The above equation can be solved by using Gaussian elimination method.

$$Let, \quad \left(\begin{array}{ccc|c} 1.25 & 0 & 0 & 10{,}000 \\ 0 & 1.89 & 0.48 & 0 \\ 0 & 0.48 & 2.026 & -12{,}000 \end{array}\right)$$

$$\Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 8{,}000 \\ 0 & 1.89 & 0.48 & 0 \\ 0 & 0.48 & 2.026 & -12{,}000 \end{array}\right) R_1 \rightarrow \frac{R_1}{1.25}$$

$$\Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 8{,}000 \\ 0 & 1.89 & 0.48 & 0 \\ 0 & 0.48 & 2.026 & -12{,}000 \end{array}\right) R_3 \rightarrow R_3 - 0.25396 R_2$$

Now, we obtained the required triangular matrix

$$\Rightarrow \quad 1 \times 10^5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.89 & 0.48 \\ 0 & 0 & 1.9041 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 8000 \\ 0 \\ -12{,}000 \end{Bmatrix}$$

$$\Rightarrow \quad 1 \times 10^5 \, (1.9041) \, u_6 = -2.5$$

$$u_6 = -0.063 \, mm$$

$$\Rightarrow \quad 1 \times 10^5 \, (1.89 \, u_5 + 0.48 \, u_6) = 0$$

$$1.89 \, u_5 = -0.48 \, (-0.063)$$

$$u_5 = 0.016 \, mm$$

$$\Rightarrow \quad 1 \times 10^5 u_3 = 8000$$

$$u_3 = 0.08 \, mm$$

Result: (1) Element stiffness matrix for each element:

$$[K]_1 = 1\times 10^5 \begin{bmatrix} 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_2 = 1\times 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.666 & 0 & -1.666 \\ 0 & 0 & 0 & 0 \\ 0 & -1.666 & 0 & 1.666 \end{bmatrix}$$

$$[K]_3 = 1\times 10^5 \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$[K]_4 = 1\times 10^5 \begin{bmatrix} 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) Global stiffness matrix:

$$\Rightarrow [K] = 1\times 10^5 \begin{bmatrix} 1.89 & 0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0 \\ 048 & 0.36 & 0 & 0 & -0.48 & -0.36 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.666 & 0 & -1.666 & 0 & 0 \\ -0.64 & -0.48 & 0 & 0 & 1.89 & 0.48 & -1.25 & 0 \\ -0.48 & -0.36 & 0 & -1.666 & 0.48 & 2.026 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) Displacements:

$$u_1 = 0$$

$$u_2 = 0$$

$$u_3 = 0.08\, mm$$

$$u_4 = 0$$

$$u_5 = 0.016\, mm$$

$$u_6 = -0.063\, mm$$

$$u_7 = 0$$

$$u_8 = 0$$

Example 2.23

For the plane truss shown in fig., determine the horizontal and vertical displacements of nodal and the stresses in each element. all elements have E=201 Gpa and A=4 × 10^{-4} m^2

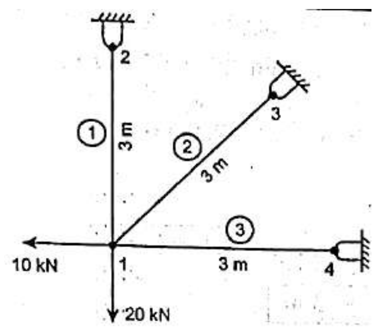


Fig. (i)

Given:

Youn's modulus, E = 201 GPa = 201 × 10^9 Pa = 201 × 10^9 N/m^2

Area of each element, A = 4 × 10^{-4} m^2

Load acting at node, 1 = - 20 kN = - 20 × 10^3 N

[∵ Load is downward direction]

Load acting at node 1 = - 10 kN = - 10 × 10^3 N

[∵ Load is acting opposite to x direction]

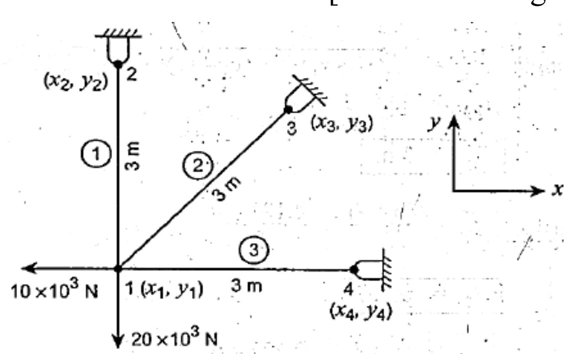


Fig. (ii)

[Node: Number with circle denotes element & Number without circle denotes Node]

To find:

(i) Nodal displacements $u_1, u_2, u_3, u_4, u_5, u_6, u_7\ and\ u_8$

(ii) Stress in each element, $\sigma_1, \sigma_2\ and\ \sigma_3$.

Solution: Consider node 1 as the origin.

The co-ordinated of various nodes are given below:

$$\text{Node 1} = (\overset{x_1}{0},\ \overset{y_1}{0})$$

$$\text{Node 2} = (\overset{x_2}{0},\ \overset{y_2}{0})$$

$$\text{Node 4} = (\overset{x_4}{3},\ \overset{y_4}{0})$$

At node 3: $x_3 = 3 \times \cos 45^o = 2.121\ m$

$$y_3 = 3 \times \sin 45^o = 2.121\ m$$

$$\text{Node 4} = (\overset{x_3}{2.121},\ \overset{y_3}{2.121})$$

For element (1): Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(0-0)^2 + (3-0)^2}$$

$$l_{e1} = 3\ mm$$

Direction consines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}} = \dfrac{0-0}{3}$

$$l_1 = 0$$

$$m_1 = \frac{y_2 - y_1}{l_{e1}} = \frac{3-0}{3}$$

$$m_1 = 1$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(2.121-0)^2 + (2.121-0)^2}$$

$$l_{e2} = 2.9995 = 3m$$

Direction consines, $l_1 = \dfrac{x_3 - x_1}{l_{e1}} \quad = \dfrac{0-0}{3}$

$$l_1 = 0$$

$$m_1 = \frac{y_2 - y_1}{l_{e1}} \quad = \frac{3-0}{3}$$

$$m_1 = 1$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(2.121-0)^2 + (2.121-0)^2}$$

$$l_{e2} = 2.9995 = 3\, m$$

Direction consines, $l_2 = \dfrac{x_3 - x_1}{l_{e2}} \quad = \dfrac{2.121-0}{3}$

$$l_3 = 0.707$$

$$m_2 = \frac{y_3 - y_1}{l_{e2}} \quad = \frac{2.121-0}{3}$$

$$m_2 = 0.707$$

For element (3): Length, $l_{e3} = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2}$

$$= \sqrt{(3-0)^2 + (0-0)^2}$$

$$l_{e3} = 3\, mm$$

Direction consines, $l_3 = \dfrac{x_4 - x_1}{l_{e2}} \quad = \dfrac{3-0}{3}$

$$l_3 = 1\, mm$$

$$m_3 = \frac{y_4 - y_1}{l_{e3}}$$

$$= \frac{0-0}{3}$$

$$m_3 = 0$$

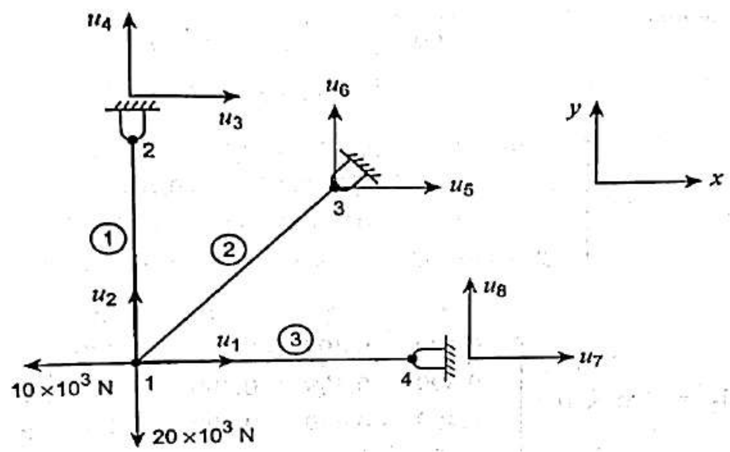


Fig. (iii)

For element (1): Displacements $u_1, u_2, u_3 and\ u_4$

Stiffness matrix [K] for a truss element is given by,

$$[K]_1 = \frac{A_1 E_1}{l_{e1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix}$$

[From equation no. (2.89)]

$$= \frac{4 \times 10^{-4} \times 201 \times 10^9}{1000} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1^2 & 0 & -(1)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1^2 \end{bmatrix}$$

$$[K]_1 = 268 \times 10^5 \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \qquad \ldots(1)$$

For element (2): Displacements $\boldsymbol{u_1, u_2, u_5 and\ u_6}$:

$$\text{Stiffness matrix } [\,\text{K}\,]_2 = \frac{A_2E_2}{l_{e2}}\begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_2^2 & -l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_2m_2 \\ -l_2m_2 & -m_2^2 & l_2m_2 & m_2^2 \end{bmatrix}$$

$$[K]_2 = 268\times10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.499 & 0.499 & -0.499 & 0.499 \\ 0.499 & 0.499 & -0.499 & -0.499 \\ -0.499 & -0.499 & 0.499 & 0.499 \\ -0.499 & -0.499 & 0.499 & 0.499 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \quad \ldots(2)$$

For element (3): Displacements $\boldsymbol{u_1, u_2, u_7 and\ u_8}$:

$$\text{Stiffness matrix } [\,\text{K}\,]_3 = \frac{A_3E_3}{l_{e3}}\begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & -l_3m_3 & -m_2^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}$$

$$= \frac{4\times10^{-4}\times201\times10^9}{3}\begin{bmatrix} (1)^2 & 0 & -(1)^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(1)^2 & 0 & 1^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_3 = 268\times10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & 0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \end{array} \quad \ldots(3)$$

Assemble the stiffness matrix [K], i.e., assemble the equation (1), (2) and (3).

	1	2	3	4	5	6	7	8	
	0 + 0.499 + 1	0 + 0.499 + 0	0	0	-0.499	-0.499	-1	0	1
[K] = 1×10^5	0 + 0.499 + 0	1 + 0.499 + 0	0	-1	-0.499	-0.499	0	0	2
	0	0	0	0	0	0	0	0	3
	0	-1	0	1	0	0	0	0	4
	-0.499	-0.499	0	0	0.499	0.499	0	0	5
	-0.499	-0.499	0	0	0.499	0.499	0	0	6
	-1	0	0	0	0	0	1	0	7
	0	0	0	0	0	0	0	0	8

$$\Rightarrow [K]$$

$$= 268 \times 10^5 \begin{bmatrix} 1.499 & 0.499 & 0 & 0 & -0.64 & -0.499 & -1 & 0 \\ 0.499 & 1.499 & 0 & -1 & -0.48 & -0.499 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ldots (4)$$

We know that, General finite element equation is

$$[K]\{u\} = \{F\}$$

$$\Rightarrow 268 \times 10^5 \begin{bmatrix} 1.499 & 0.499 & 0 & 0 & -0.64 & -0.499 & -1 & 0 \\ 0.499 & 1.499 & 0 & -1 & -0.48 & -0.499 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -0.499 & -0.499 & 0 & 0 & 0.499 & 0.499 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} \qquad \dots (5)$$

Applying boundary conditions [Refer Fig. (iii)]:

(i) Node 2 is fixed. So, $u_3 = u_4 = 0$

(ii) Node 3 is fixed. So, $u_5 = u_6 = 0$.

(iii) Node 4 fixed. So, $u_7 = u_8 = 0$

(iv) At node 1, point load of 20×10^3N is acting in downward direction. So, F=-20 10^3 N.

(v) At node1, point load of 20×10^3N is acting opposite to *x* direction. So, F=-10 $\times$ 10^3 N.

(vi) Self-weight is neglected. So, F3=F4=F5=F6=F7=F8=0.

Substitute the above boundary condition values in equation no.(5), $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$. So, delete third row third column, fourth row fourth column, fifth row fifth column, sixth row sixth column, seventh row seventh column, and eighth row eighth column of [K] matrix.

The final reduced equation is,

$$268 \times 10^5 \begin{bmatrix} 1.499 & 0.499 \\ 0.499 & 1.499 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow 268 \times 10^5 \begin{bmatrix} 1.499 & 0.499 \\ 0.499 & 1.499 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -10 \times 10^3 \\ -20 \times 10^3 \end{Bmatrix}$$

$$\Rightarrow 268 \times 10^5 (1.499\, u_1 + 0.499\, u_2) = -10 \times 10^3 \qquad \text{... (6)}$$

$$268 \times 10^5 (0.499\, u_1 + 1.499\, u_2) = -20 \times 10^3 \qquad \text{... (7)}$$

Equation (7) × 3.005 ⇒

$$268 \times 10^5 (1.499\, u_1 + 4.504\, u_2) = -60.1 \times 10^3 \qquad \text{... (8)}$$

Solving equation (6) and (8),

$$\Rightarrow 268 \times 10^5 (1.499\, u_1 + 0.499\, u_2) = -10 \times 10^3$$

$$268 \times 10^5 (1.499\, u_1 + 4.504\, u_2) = -60.1 \times 10^3$$

$$268 \times 10^5 (-4.005\, u_2) = -50.1 \times 10^3$$

$$\Rightarrow u_2 = -4.66 \times 10^{-4} m$$

Substitute u_2 value in equation (6).

$$268 \times 10^5 (1.499\, u_1 + 0.499(-4.66 \times 10^{-4}) = -10 \times 10^3$$

$$\Rightarrow u_1 = -9.379 \times 10^{-5} m$$

We know that,

$$Stress, \sigma = \frac{E}{l_e} [-l \quad -m \quad l \quad m] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

For element (1):

$$Stress, \sigma_1 = \frac{E}{l_e} [-l_1 \quad -m_1 \quad l_1 \quad m_1] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{201 \times 10^9}{1000} [0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} -9.379 \times 10^5 \\ -4.66 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 67 \times 10^9 \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -9.379 \times 10^5 \\ -4.66 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 67 \times 10^9 [0 + 4.66 \times 10^{-4} + 0 + 0$$

$$\sigma_1 = 31.22 \times 10^6 N/m^2$$

For element (2):

$$Stress, \sigma_2 = \frac{E_2}{l_{e2}} \begin{bmatrix} -l_2 & -m_2 & l_2 & m_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= \frac{201 \times 10^9}{1000} \begin{bmatrix} -7.07 & -0.707 & 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} -9.379 \times 10^5 \\ -4.66 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 67 \times 10^9 [-0.707 \times (9.379 \times 10^{-5}) + (-0.707) \times (-4.66 \times 10^{-4}) + 0 + 0]$$

$$\sigma_2 = 26.52 \times 10^6 N/m^2$$

For element (3):

$$Stress, \sigma_3 = \frac{E_3}{l_{e3}} \begin{bmatrix} -l_3 & -m_3 & l_3 & m_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$= \frac{201 \times 10^9}{1000} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -9.379 \times 10^5 \\ -4.66 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 67 \times 10^9 [-9.379 \times 10^{-5} + (-1) + 0 + 0 + 0]$$

$$\sigma_3 = 6.28 \times 10^6 N/m^2$$

Result: (i) Nodal displacements:

$$u_1 = -9.379 \times 10^{-5} m$$

$$u_2 = -4.66 \times 10^{-4} m$$

$$u_3 = 0$$

$$u_4 = 0$$

$$u_5 = 0$$

$$u_6 = 0$$

$$u_7 = 0$$

$$u_8 = 0$$

(ii) Stresses in each element:

$$\sigma_1 = 31.22 \times 10^6 N/m^2$$

$$\sigma_2 = 26.52 \times 10^6 N/m^2$$

$$\sigma_3 = 6.28 \times 10^6 N/m^2$$

Example 2.24

For the three-bar truss shown in fig (i), determine the displacements of node 1 and the stress in element 3.

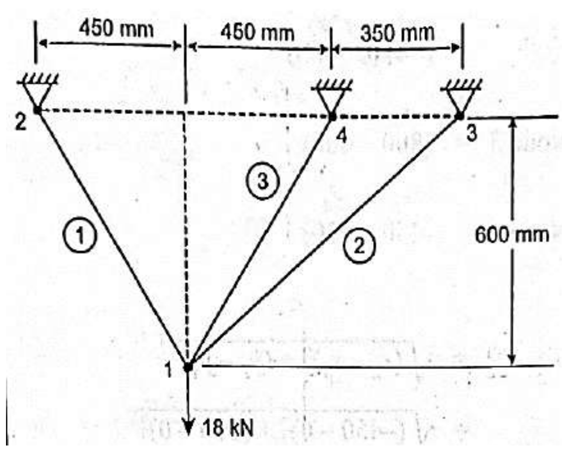


Fig (i)

Given:

Area of cross section, A of each member} = 250 mm^2

Youn's modulus, E = 200 GPa = 200 × 10^9 Pa = 200 × 10^9 N/m^2

$= 200 \times 10^3$ N/m^2

Area of each element, A = 4×10^{-4} m^2

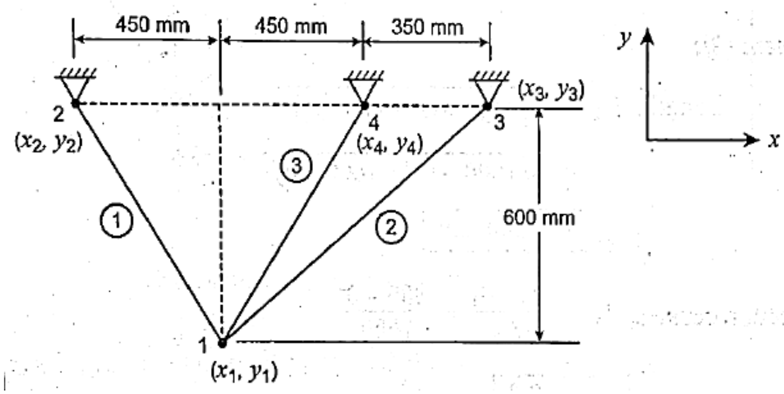


Fig. (ii)

To find:

1. Displacements of node 1.
2. Stress in each element, 3.

Solution: Consider node 1 as the origin.

The co-ordinated of various nodes are given below:

$$\text{Node } 1 = \overset{x_1}{(0,}\ \overset{y_1}{0)}$$

$$\text{Node } 2 = \overset{x_2}{(-450,}\ \overset{y_2}{600)}$$

$$\text{Node } 3 = \overset{x_3}{(800,}\ \overset{y_3}{600)}$$

$$\text{Node } 4 = \overset{x_4}{(450,}\ \overset{y_4}{600)}$$

For element (1): Length, $l_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(-450 - 0)^2 + (600 - 0)^2}$$

$$l_{e1} = 750\ mm$$

Direction consines, $l_1 = \dfrac{x_2 - x_1}{l_{e1}} \quad = \dfrac{-450 - 0}{750}$

$$l_1 = -0.6$$

$$m_1 = \frac{y_2 - y_1}{l_{e1}} \quad = \frac{600 - 0}{750}$$

$$m_1 = 0.8$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(800 - 0)^2 + (600 - 0)^2}$$

$$l_{e2} = 1000\, m$$

Direction consines, $l_1 = \dfrac{x_3 - x_1}{l_{e1}} \quad = \dfrac{450 - 0}{750}$

$$l_1 = -0.6$$

$$m_1 = \frac{y_2 - y_1}{l_{e1}} \quad = \frac{600 - 0}{3}$$

$$m_1 = 0.8$$

For element (2): Length, $l_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

$$= \sqrt{(800 - 0)^2 + (600 - 0)^2}$$

$$l_{e2} = 1000\, m$$

Direction consines, $l_2 = \dfrac{x_3 - x_1}{l_{e2}} \quad = \dfrac{800 - 0}{1000}$

$$l_3 = 0.8$$

$$m_2 = \frac{y_3 - y_1}{l_{e2}} \quad = \frac{600 - 0}{1000}$$

$$m_2 = 0.6$$

For element (3): Length, $l_{e3} = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2}$

$$= \sqrt{(450 - 0)^2 + (600 - 0)^2}$$

$$l_{e3} = 750\ mm$$

Direction consines, $l_3 = \dfrac{x_4 - x_1}{l_{e2}} = \dfrac{450 - 0}{750}$

$$l_3 = 0.6\ mm$$

$$m_3 = \frac{y_4 - y_1}{l_{e3}} = \frac{600 - 0}{750}$$

$$m_3 = 0.8$$

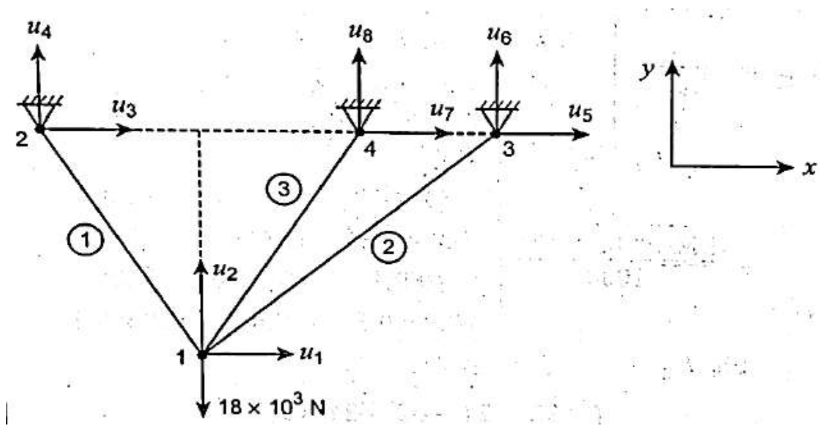


Fig. (iii)

For element (1): Displacements $u_1, u_2, u_3 and\ u_4$

Stiffness matrix [K] for a truss element is given by,

$$[K]_1 = \frac{A_1E_1}{l_{e1}} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$= \frac{250 \times 200 \times 10^3}{750} \begin{bmatrix} (-0.6)^2 & -0.6 \times 0.8 & -(-0.6)^2 & -0.6 \times 0.8 \\ 0.6 \times 0.8 & (0.8)^2 & 0.6 \times 0.8 & -(0.8)^2 \\ (-0.6)^2 & 0.6 \times 0.8 & (-0.6)^2 & -0.6 \times 0.8 \\ 0.6 \times 0.8 & -(0.8)^2 & -0.6 \times 0.8 & (0.8)^2 \end{bmatrix}$$

$$[K]_1 = 1 \times 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 24 & -32 & -24 & 32 \\ -32 & 42.666 & 32 & -42.666 \\ -24 & 32 & 24 & -32 \\ 32 & -42.666 & -32 & 42.666 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \ldots(1)$$

For element (2): Displacements u_1, u_2, u_5 and u_6:

$$\text{Stiffness matrix } [\,\mathrm{K}\,]_2 = \frac{A_2E_2}{l_{e2}} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} l_2^2 & l_2m_2 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_2^2 & -l_2m_2 & -m_2^2 \\ -l_2^2 & -l_2m_2 & l_2^2 & l_2m_2 \\ -l_2m_2 & -m_2^2 & l_2m_2 & m_2^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$= \frac{250 \times 200 \times 10^3}{1000} \begin{bmatrix} (0.8)^2 & -0.6 \times 0.8 & -(0.8)^2 & -0.8 \times 0.6 \\ 0.8 \times 0.6 & (0.6)^2 & -0.8 \times 0.6 & -(0.6)^2 \\ -(0.8)^2 & -0.8 \times 0.6 & (0.8)^2 & 0.8 \times 0.6 \\ -0.8 \times 0.6 & -(0.6)^2 & -0.6 \times 0.8 & (0.6)^2 \end{bmatrix}$$

$$[K]_2 = 1 \times 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 32 & 24 & -32 & -24 \\ 24 & 18 & -24 & -18 \\ -32 & -24 & 32 & 24 \\ -24 & -18 & 24 & 18 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \qquad \ldots(2)$$

For element (3): Displacements $u_1, u_2, u_7 and$ u_8:

$$\text{Stiffness matrix } [\,\mathrm{K}\,]_3 = \frac{A_3E_3}{l_{e3}} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ l_3m_3 & m_3^2 & -l_3m_3 & -m_2^2 \\ -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \end{array}$$

$$= \frac{250 \times 200 \times 10^3}{750} \begin{bmatrix} (0.6)^2 & 0.6 \times 0.8 & -(0.6)^2 & -0.6 \times 0.8 \\ 0.6 \times 0.8 & (0.8)^2 & -0.6 \times 0.8 & -(0.8)^2 \\ -(0.6)^2 & -0.6 \times 0.8 & (0.6)^2 & 0.6 \times 0.8 \\ -0.6 \times 0.8 & -(0.8)^2 & 0.6 \times 0.8 & (0.8)^2 \end{bmatrix}$$

$$[K]_3 = 268 \times 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 24 & 32 & -24 & -32 \\ 32 & 42.666 & -32 & -42.666 \\ -24 & -32 & 24 & 32 \\ -32 & -42.666 & 32 & 42.666 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array} \qquad \ldots(3)$$

Assemble the stiffness matrix [K], i.e., assemble the equation (1), (2) and (3).

[K] = 1 × 10^5

1	2	3	4	5	6	7	8	
24 + 32 + 24	-32 + 24 + 32	-24	32	-32	-24	-24	-32	1
-32 + 24 + 32	42.666 + 18 + 42.666	32	-42.666	-24	-18	-32	-42.66	2
-24	32	24	-32	0	0	0	0	3
32	-42.666	-32	42.666	0	0	0	0	4
-32	-24	0	0	32	24	0	0	5
-24	-18	0	0	24	18	0	0	6
-24	-32	0	0	0	0	24	32	7
-32	-42.666	0	0	0	0	32	42.666	8

$$\Rightarrow [K] = 1 \times 10^3 \begin{bmatrix} 80 & 24 & -24 & 32 & -32 & -24 & -24 & -32 \\ 24 & 103.332 & 32 & -42.666 & -24 & -18 & -32 & -42.666 \\ -24 & 32 & 24 & -32 & 0 & 0 & 0 & 0 \\ 32 & -42.666 & -32 & 42.666 & 0 & 0 & 0 & 0 \\ -32 & -24 & 0 & 0 & 32 & 24 & 0 & 0 \\ -24 & -18 & 0 & 0 & 24 & 18 & 0 & 0 \\ -24 & -32 & 0 & 0 & 0 & 0 & 24 & 32 \\ -32 & -42.666 & 0 & 0 & 0 & 0 & 32 & 42.666 \end{bmatrix}$$

We know that, General finite element equation is

$$[K]\{u\} = \{F\}$$

$$\Rightarrow 1 \times 10^3 \begin{bmatrix} 80 & 24 & -24 & 32 & -32 & -24 & -24 & -32 \\ 24 & 103.332 & 32 & -42.666 & -24 & -18 & -32 & -42.666 \\ -24 & 32 & 24 & -32 & 0 & 0 & 0 & 0 \\ 32 & -42.666 & -32 & 42.666 & 0 & 0 & 0 & 0 \\ -32 & -24 & 0 & 0 & 32 & 24 & 0 & 0 \\ -24 & -18 & 0 & 0 & 24 & 18 & 0 & 0 \\ -24 & -32 & 0 & 0 & 0 & 0 & 24 & 32 \\ -32 & -42.666 & 0 & 0 & 0 & 0 & 32 & 42.666 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} \qquad \ldots (4)$$

Applying boundary conditions [Refer Fig. (iii)]:

1. Node 2 is fixed. So, $u_3 = u_4 = 0$
2. Node 4 is fixed. So, $u_7 = u_8 = 0$.
3. Node 3 fixed. So, $u_5 = u_6 = 0$
4. At node 1, point load of 18×10^3N is acting in downward direction.

   So, $F=-18\times10^3$ N.
5. Self-weight is neglected. So, $F_1=F_3=F_4=F_5=F_6=F_7=F_8=0$.

Substitute the above values in equation (4), $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$ delete third row third column, fourth row fourth column, fifth row fifth column, sixth row sixth column, seventh row seventh column and eighth row eighth column of [K] matrix. Hence the final reduced equation is,

$$1 \times 10^3 \begin{bmatrix} 80 & 24 \\ 24 & 103.332 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow 1 \times 10^3 \begin{bmatrix} 80 & 24 \\ 24 & 103.332 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -18 \times 10^3 \end{Bmatrix}$$

$$\Rightarrow \quad 80\, u_1 + 24\, u_2 = 0 \qquad \ldots (5)$$

$$24\, u_1 + 103.332\, u_2 = 0 \qquad \ldots (6)$$

Equation (6) × 3.3334 ⇒

$$\Rightarrow \quad 80\, u_1 + 344.44\, u_2 = -60 \qquad \ldots (7)$$

$$80\, u_1 + 24\, u_2 = 0 \qquad \ldots (5)$$

Solving, $\quad 320.44\, u_2 = 0$

$$\Rightarrow u_2 = -0.187\, mm$$

Substitute u_2 value in equation (5).

$$\Rightarrow \quad 80\, u_1 + 24\, (-0.187 = -0$$

$$\Rightarrow \quad u_1 = 0.0561\, mm$$

We know that,

$$Stress, \sigma = \frac{E}{l_e} [-l \quad -m \quad l \quad m] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

For element (3):

$$Stress, \sigma_3 = \frac{E_3}{l_{e3}} [-l_3 \quad -m_3 \quad l_3 \quad m_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$= \frac{200 \times 10^3}{750} [-0.6 \quad -0.8 \quad 0.6 \quad 0.8] \begin{Bmatrix} 0.561 \\ -0.187 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 266.6667[-0.6 \times 0.561 + (-0.8) \times (-0.187) + 0 + 0]$$

$$\sigma_3 = -49.86\ N/m^2$$

Result: (i) Displacements of node 1:

$$u_1 = 0.561\ mm$$

$$u_2 = -0.187\ mm$$

(ii) Stresses in element(3):

$$\sigma_3 = -49.86 N/m^2$$

2.12. BENDING Of BEAMS

2.12.1. Introduction

Beam is a structural member which is supported along the length and subjected to external forces or loads acting transversely *i.e.,* perpendicular to the centre line. Beam is sufficiently long when compared to the lateral dimensions.

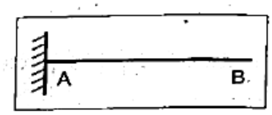


Fig. 2.29(a)

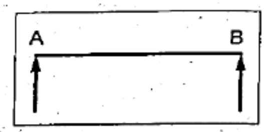


Fig. 2.29(b)

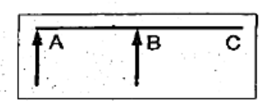


Fig. 2.29(c)

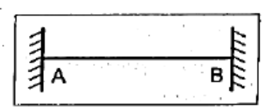


Fig. 2.29(d)

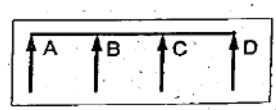


Fig. 2.29(e)

2.12.2. Types of Beams

The following are the important types of beams.

(a) Cantilever beam

(b) Simply supported beam

(c) Overhanging beam

(d) Fixed beam and

(e) Continuous beam

(a) Cantilever Beam

A beam with one end free and the other end fixed is called cantilever beam. Refer Fig.2.29(a).

(b) Simply Supported Beam (SSB)

A beam supported or resting freely on the supports at its both ends is called SSB. It is shown in Fig.2.29(b).

(c) Overhanging Beam

If the one or both the end portions are extended beyond the support, then it is called overhanging beam. Refer Fig.2.29(c).

(d) Fixed Beam

A beam whose both ends are fixed or built into the walls, is called a fixed beam. Refer Fig.2.29(d).

(e) Continues beam

A beam which has more than two supports is called continuous beam. Refer Fig.2.29(e)

2.12.3. Transverse Loading on Beams

A load which is acting vertically downward on the horizontal beam is called transverse load.

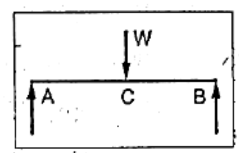


Fig. 2.30(a)

2.13.4. Types of Transverse Load

A beam may be subjected to the following types of loads.

(a) Point or concentrated load.

(b) Uniformly Distributed Load (UDL)

(c) uniformly Varying Load (UVL)

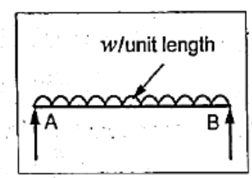


Fig. 2.30(b)

(a) Point or Concentrated Load

A load (W) which is acting at a particular point is called point load. Refer Fig.2.30(a).

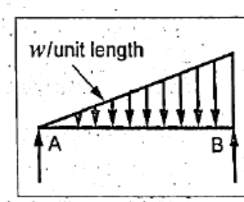


Fig. 2.30(c)

(b) Uniformly Distributed Load (UDL)

A load which is spread over a beam in such a manner that the rate of loading 'w' is uniform throughout the length. Refer Fig.2.30(b).

(c) Uniformly Varying Load (UVL)

A load which is spread over a beam in such a manner that the rate of loading uniformly varies from point to point along the beam. The load is zero at one end and increases uniformly to the other end. It is also called as triangular load. Refer Fig.2.30(c).

2.12.5. Derivation of shape Function for Beam Element [Fourth Order Beam Equation]

Consider the beam element as shown in Fig.2.31. The beam is of length L with axial local co-ordinate x and transverse local co-ordinate y. The local transverse nodal displacements are given by d_{ly} and d_{2y}' The rotations are given by $1 and $2' The local nodal forces are given by Fly and F 2y- The bending moments are given by ml and m.

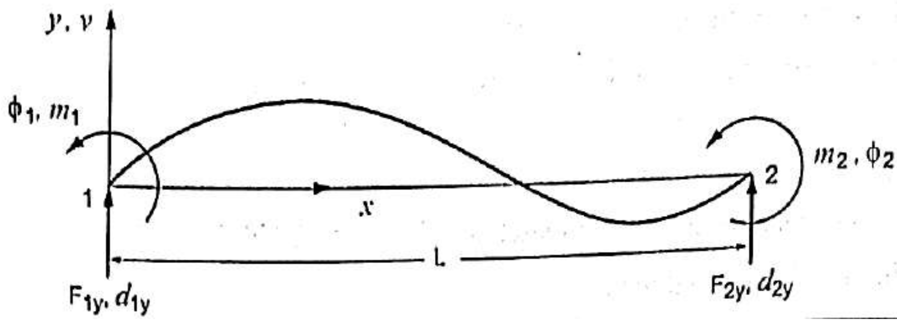


Fig. 2.31. Beam element with positive nodal displacements, rotations, forces, and moments

At all nodes, the following sign conversions are used.

(i) Moments are positive in the counterclockwise direction.

(ii) Rotations are positive in the counterclockwise direction.

(iii) Forces are positive in the positive y direction

(iv) Displacements are positive in the positive y direction.

Fig.2.32 indicates the sign conventions used in simple beam theory for positive shear forces F and bending moments m

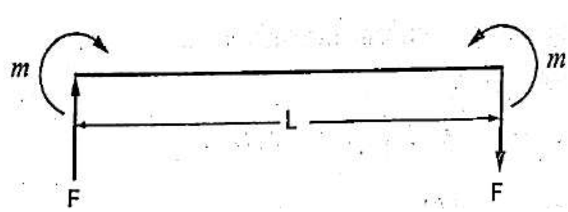


Fig. 2.32 Beam theory sign conventions for shear forces and bending moments

Assume the transverse displacement variation through the element length to be

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \qquad \text{... (2.100)}$$

We express *v* in terms of the nodal degrees of freedom $d_{1y}, d_{2y}\phi_1\ and\ \phi_2$ as follows:

At $x = 0$,

$$v(0) = a_4 = d_{1y} \qquad \text{... (2.101)}$$

$$\frac{dv_{(x)}}{dx} = 3a_1x^2 + 2a_2x + a_3$$

$$\frac{dv_{(0)}}{dx} = a_3 = \phi_1 \qquad \text{... (2.102)}$$

When $x = L$,

$$v(L) = a_1L^3 + a_2L^2 + a_3L + a_4 = \ d_{2y} \qquad \text{... (2.103)}$$

$$\frac{dv_{(L)}}{dx} = 3a_1L^2 + 2a_2L + a_3 = \phi_2 \qquad \text{... (2.104)}$$

Finding a_1 and a_2 in terms of $d_{1y}, d_{2y}\phi_1\ and\ \phi_2$ by using the above equations (2.101), (2.102), (2.103) and (2.104).

$$(2.103) \Rightarrow \qquad d_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$= a_1L^3 + a_2L^2 + a_3L + d_{1y} \qquad [\because a_4 = d_{1y}]$$

$$\Rightarrow \qquad (d_{2y} - d_{1y}) = a_1L^3 + a_2L^2 + a_3L + \phi_1L$$

$$\Rightarrow (d_{2y} - d_{1y} - \phi_1L\ ) = a_1L^3 + a_2L^2$$

$$\Rightarrow \frac{1}{L}\left(d_{2y} - d_{1y} - \phi_1L\right) = a_1L^3 + a_2L \qquad \text{... (2.105)}$$

$$(2.104) \Rightarrow \qquad \phi_2 = 3a_1L^2 + 2a_2L + a_3$$

$$= 3a_1L^2 + 2a_2L + \phi_1 \qquad [\because a_3 = \phi_1]$$

$$\Rightarrow \qquad \phi_2 - \phi_1 = 3a_1L^2 + 2a_2L \qquad \text{... (2.106)}$$

Equation (2.105) × 3

$$\Rightarrow \frac{3}{L}\left(d_{2y} - d_{1y} - \phi_1L\right) = 3a_1L^2 + 3a_2L \qquad \text{... (2.107)}$$

Solving equation (2.106) and (2.107)

$$\phi_2 - \phi_1 = 3a_1L^2 + 2a_2L$$

$$\frac{3}{L}\left(d_{2y} - d_{1y} - \emptyset_1 L\right) = 3a_1 L^2 + 3a_2 L$$

Subtracting, $\quad \phi_2 - \phi_1 - \dfrac{3}{L}\left(d_{2y} - d_{1y} - \emptyset_1 L\right) = -a_2 L$

$$\phi_2 - \phi_1 - \frac{3}{L}\left(d_{2y} - d_{1y}\right) + \frac{3}{L}\emptyset_1 L = -a_2 L$$

$$\phi_2 - \phi_1 - \frac{3}{L}\left(d_{2y} - d_{1y}\right) + 3\emptyset_1 = -a_2 L$$

$$\phi_2 - 2\phi_1 - \frac{3}{L}\left(d_{2y} - d_{1y}\right) = -a_2 L$$

$$\frac{1}{L}(\phi_2 + 2\phi_1) - \frac{3}{L^2}\left(d_{2y} - d_{1y}\right) = -a_2$$

$$\Rightarrow \quad \frac{1}{L}(\phi_2 + 2\phi_1) + \frac{3}{L^2}\left(d_{2y} - d_{1y}\right) = -a_2$$

$$\Rightarrow \quad \frac{-1}{L}(\phi_2 + 2\phi_1) - \frac{3}{L^2}\left(d_{1y} - d_{2y}\right) = a_2$$

$$\Rightarrow \quad a_2 = \frac{-3}{L}\left(d_{1y} - d_{2y}\right) - \frac{1}{L}(2\phi_1 + \phi_2)$$

…(2.108)

Substitute a_2 value in equation (2.104)

$$\Rightarrow \quad \phi_2 = 3a_1 L^2 + 2L\left[\frac{-3}{L}\left(d_{1y} - d_{2y}\right) - \frac{1}{L}(2\phi_1 + \phi_2)\right] + a_3$$

$$= 3a_1 L^2 - \frac{6}{L}\left(d_{1y} - d_{2y}\right) - 2(2\phi_1 + \phi_2) + \phi_1$$

$[\because a_3 = \phi_1]$

$$\Rightarrow \quad \phi_2 - \phi_1 = 3a_1 L^2 - \frac{6}{L}\left(d_{1y} - d_{2y}\right) - 4\phi_1 - 2\phi_2$$

$$\Rightarrow \quad 3\phi_1 + 3\phi_2 = 3a_1 L^2 - \frac{6}{L}\left(d_{1y} - d_{2y}\right)$$

$$\Rightarrow \quad 3a_1 L^2 = 3\phi_1 + 3\phi_2 + \frac{6}{L}\left(d_{1y} - d_{2y}\right)$$

$$a_1L^2 = \phi_1 + 3\phi_2 + \frac{2}{L}\left(d_{1y} - d_{2y}\right)$$

$$\Rightarrow \qquad a_1 = \frac{1}{L^2}(\phi_1 + \phi_2) + \frac{2}{L^2}\left(d_{1y} - d_{2y}\right)$$

$$\Rightarrow \qquad a_1 = \frac{2}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2) \qquad \text{... (2.109)}$$

Substitute a_1, a_2, a_3 *and* a_4 value in equation (2.100)

$$v(x) = \left[\frac{2}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L}\left(d_{1y} - d_{2y}\right) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y} \qquad \text{... (2.110)}$$

$$[\because a_3 = \phi_1; a_4 = d_{1y}]$$

In matrix form, $v(x) = [\ N\ ]\{\ d\ \}$

$$\Rightarrow \quad v(x) = [N_1 \ \ N_2 \ \ N_3 \ \ N_4]\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$\Rightarrow \quad v(x) = N_1 d_{1y} + N_2\phi_1 + N_3 d_{2y} + N_4\phi_2 \qquad \text{... (2.111)}$$

Where N_1, N_2, N_3 and N_4 are shape functions for beam element.

$$\left.\begin{aligned} N_1 &= \frac{1}{L^3}(2x^3 - 3x^2L + L^2) \\ N_2 &= \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \\ N_3 &= \frac{1}{L^3}(-2x^3 + 3x^2L) \\ N_4 &= \frac{1}{L^3}(x^3L - x^2L^2) \end{aligned}\right\} \qquad \text{... (2.112)}$$

Verification: we know

$$v(x) = N_1 d_{1y} + N_2\phi_1 + N_3 d_{2y} + N_4\phi_2 \qquad \text{... (2.111)}$$

Substituting N_1, N_2, N_3 and N_4 values,

$$\Rightarrow \quad v(x) = \frac{1}{L^3}(2x^3 - 3x^2L + L^2)d_{1y} + \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3)\phi_1 + \frac{1}{L^3}(-2x^3 + 3x^2L)d_{2y} + \frac{1}{L^3}(x^3L - x^2L^2)\phi_2$$

$$\Rightarrow \quad v(x) = \frac{2x^3}{L^3}d_{1y} - \frac{3x^2L}{L^3}d_{1y} + d_{2y} + \frac{x^3L}{L^3}\phi_1 - \frac{2x^2L^2}{L^3}\phi_1 + x\phi_1 - \frac{2x^3}{L^3}d_{2y} + \frac{3x^2L}{L^3}d_{2y} + \frac{x^3L\phi_2}{L^3} - \frac{x^2L^2}{L^3}\phi_2$$

$$v(x) = \frac{2x^3}{L^3}d_{1y} - \frac{3x^2}{L^2}d_{1y} + d_{2y} + \frac{x^3}{L^2}\phi_1 - \frac{2x^2}{L}\phi_1 + x\phi_1 - \frac{2x^3}{L^3}d_{2y} + \frac{3x^2}{L^2}d_{2y} + \frac{x^3\phi_2}{L^2} - \frac{x^2\phi_2}{L}$$

$$\Rightarrow \quad v(x) = \left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y}$$

[Same as equation no. (2.110)]

2.12.6. Stiffness Matrix [K] for beam element

The stiffness matrix [k] for beam element is derived by using a direct equilibrium approach and beam theory sign conversions.

We know that,

Transverse displacement

$$v(x) = \left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y}$$

$$\Rightarrow \quad \frac{dv_{(x)}}{dx} = 3x^2\left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + 2x\left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right] + \phi_1$$

$$\Rightarrow \quad \frac{d^2 v_{(x)}}{dx^2} = 6x\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + 2\left[\frac{-3}{L^2}\left(d_{1y} - d_{2y}\right) - \frac{1}{L}(2\phi_1 + \phi_2)\right] \quad \text{... (2.13)}$$

$$\Rightarrow \quad \frac{d^2 v_{(x)}}{dx^3} = 6\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] \quad \text{... (2.14)}$$

Put $x = 0$ in equation (2.113)

$$\Rightarrow \quad \frac{d^2 v_{(0)}}{dx^3} = 0 + 2\left[\frac{-3}{L^3}\left(d_{1y} - d_{2y}\right) - \frac{1}{L}(2\phi_1 + \phi_2)\right]$$

$$= \frac{-6}{L^2}\left(d_{1y} - d_{2y}\right) - \frac{2}{L}(2\phi_1 + \phi_2)$$

$$= \frac{1}{L^2}\left[-6Ld_{1y} + 6Ld_{2y} - 4L^2\phi_1 - 2L^2\phi_2\right] \quad \text{... (2.115)}$$

Put $x = \mathrm{L}$ in equation (2.113)

$$\Rightarrow \quad \frac{d^2 v(L)}{dx^3} = 6L\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + 2\left[\frac{-3}{L^3}\left(d_{1y} - d_{2y}\right) - \frac{1}{L}(2\phi_1 + \phi_2)\right]$$

$$= \frac{12L}{L^2}\left(d_{1y} - d_{2y}\right) + \frac{6L}{L}(\phi_1 + \phi_2) - \frac{6}{L^2}\left(d_{1y} - d_{2y}\right) - \frac{2}{L}(2\phi_1 + \phi_2)$$

$$= \frac{1}{L^3}\left[12Ld_{1y} - 12Ld_{2y} + 6L^2\phi_1 + 6L^2\phi_2 - 6Ld_{1y} + 6Ld_{2y} - 4L^2\phi_1 - 2L^2\phi_2\right]$$

$$\frac{d^2 v(L)}{dx^3} = \frac{1}{L^3}\left[6Ld_{1y} + 2Ld_{2y} - \ 6Ld_{2y} + 4L^2\phi_2\right] \quad \text{... (2.116)}$$

Put $x = 0$ in equation (2.114)

$$\Rightarrow \quad \frac{d^3 v(0)}{dx^3} = 6\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]$$

$$= \frac{1}{L^3}\left[12d_{1y} - 12d_{2y} - \ 6L\phi_1 + 4L^2\phi_2\right]$$

$$\frac{d^3v(0)}{dx^3} = \frac{1}{L^3}\left[12d_{1y} + 6L\phi_1 - 12d_{2y} + 6L\phi_2\right] \qquad \text{... (2.117)}$$

Put x = L in equation (2.113)

$$\Rightarrow \quad \frac{d^3v(L)}{dx^3} = 6\left[\frac{2}{L^3}\left(d_{1y} - d_{2y}\right) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]$$

$$= \frac{1}{L^3}\left[12d_{1y} - 12d_{2y} + 6L\phi_1 - 6L^2\phi_2\right]$$

$$\frac{d^3v(L)}{dx^3} = \frac{1}{L^3}\left[12d_{1y} + 6L\phi_1 - 12d_{2y} + 6L\phi_2\right] \qquad \text{... (2.118)}$$

We know that,

$$\text{Nodal force}, F_{1y} = EI\frac{d^3v(0)}{dx^3}$$

$$\Rightarrow \qquad F_{1y} = \frac{EI}{L^3}\left[12d_{1y} + 6L\phi_1 - 12d_{2y} + 6L\phi_2\right]$$

[From equation no. (2.117)]

$$\text{Bending moment}, m_1 = -EI\frac{d^2v(0)}{dx^2}$$

$$= -\frac{EI}{L^3}\left[-6Ld_{1y} - 4L^2\phi_1 + 6Ld_{2y} - 2L^2\phi_2\right]$$

[From equation no. (2.115)]

$$m_1 = \frac{EI}{L^3}\left[6Ld_{1y} + 4L^2\phi_1 - 6Ld_{2y} + 2L^2\phi_2\right]$$

$$\text{Nodal force}, F_{2y} = -EI\frac{d^3v(0)}{dx^3}$$

$$\Rightarrow \qquad F_{2y} = \frac{-EI}{L^3}\left[12d_{1y} + 6L\phi_1 - 12d_{2y} + 6L\phi_2\right]$$

[From equation no. (2.118)]

$$\Rightarrow \qquad F_{2y} = \frac{EI}{L^3}\left[-12d_{1y} - 6L\phi_1 + 12d_{2y} - 6L\phi_2\right]$$

$$\text{Bending moment}, m_2 = -EI\frac{d^2v(L)}{dx^2}$$

$$\Rightarrow \qquad m_2 = \frac{EI}{L^3}\left[6Ld_{1y} + 2L^2\phi_1 - 6Ld_{2y} + 4L^2\phi_2\right]$$

[From equation no. (2.116)]

Arranging to the above equation $(F_{1y}, m_1, F_{2y}, m_2)$ in matrix form,

$$\Rightarrow \quad \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} \qquad \text{... (2.120)}$$

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

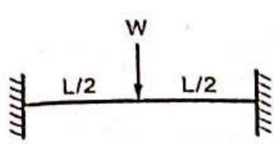
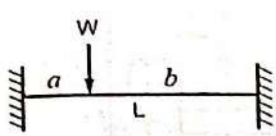
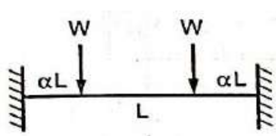
Where E = Young's modulus

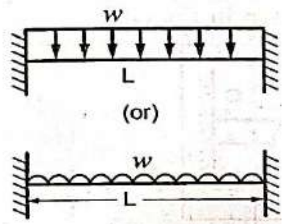
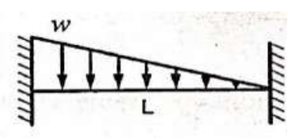
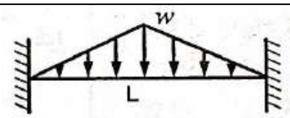
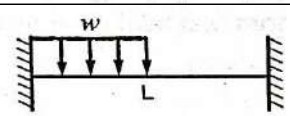
I = Moment of inertia

L = Length of the beam

2.12.7. Nodal Forces and Bending Moments

The nodal forces (F_{1y} and F_{2y}) and bending moments (m_1 and m_2) for different types of loads on fixed beams are given in Table 2.1

	Loading case	F_{1y}	m_1	F_{2y}	m_2
1.		$-\frac{W}{2}$	$-\frac{WL}{8}$	$-\frac{W}{2}$	$\frac{WL}{8}$
2.		$\frac{-Wb^2(L+2a)}{L^3}$	$\frac{-Wab^2}{L^2}$	$\frac{-Wb^2(L+2a)}{L^3}$	$\frac{Wa^2b}{L^2}$
3.		-W	$-\alpha(1 - \alpha)WL$	-W	$\alpha(1 - \alpha)WL$

4.		$\frac{-wL}{2}$	$\frac{-wL^2}{12}$	$\frac{wL}{2}$	$\frac{wL^2}{12}$
5.		$\frac{-7wL}{20}$	$\frac{-wL^2}{20}$	$\frac{-3wL}{20}$	$\frac{wL^2}{30}$
6.		$\frac{-wL}{4}$	$\frac{-5wL^2}{96}$	$\frac{-wL}{4}$	$\frac{5wL^2}{96}$
7.		$\frac{-13wL}{32}$	$\frac{-11wL^2}{192}$	$\frac{-3wL}{32}$	$\frac{5wL^2}{192}$

2.13.8. Solved Problems

EXAMPLE 2.25

A fixed beam of length 2 L m carries a uniformly distributed load of w (N/m) which run over a length of L m from the fixed end, as shown in Fig.(i). Calculate the rotation at point B.

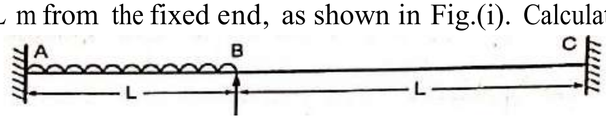


Fig.(i)

Given:

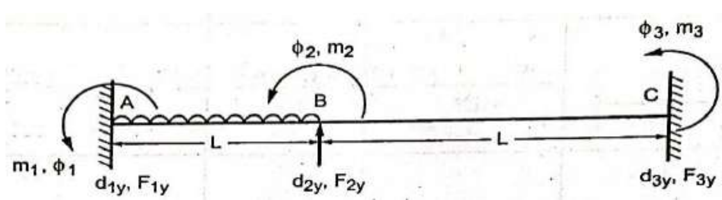


Fig.(ii)

To find: Rotation at point B, ϕ_2.

Solution: we can divide the beam into two elements as shown in Fig. (iii)

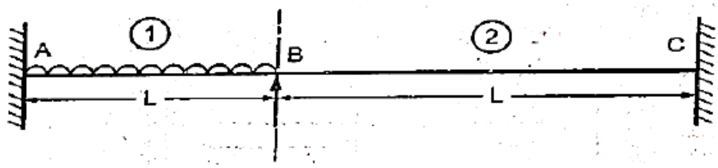


Fig.(iii)

For element (1) (Nodes 1, 2, 3, 4 i.e., $d_{1y}, \phi_1, d_{2y}, \phi_2$

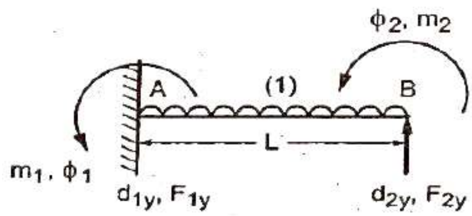


Fig.(iv)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ F_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} \qquad \ldots (1)$$

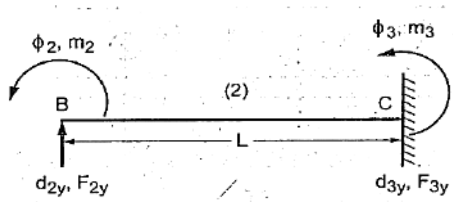
For uniformly distributed load,

$$F_{1y} = \frac{-wL}{2}; \quad m_1 = \frac{-wL^2}{12}; \; F_{2y} = \frac{-wL}{2}; \quad m_2 = \frac{wL^2}{12}$$

[Refer table 2.1]

$$(1) \Rightarrow \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ F_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-wL}{2} \\ \dfrac{-wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} \qquad \ldots (2)$$

For element (2) (Nodes 3, 4, 5, 6 i.e., $d_{2y}, \phi_2, d_{3y}, \phi_3$



Fig(iv)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ F_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_4 \end{Bmatrix} \qquad \text{... (3)}$$

There is no load and moment on element (2). So, $F_{2y} = F_{3y} = 0$ and $m_2 = m_3 = 0$.

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ F_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{... (4)}$$

Assemble the finite elements, i.e., assemble the equations (2) and (4),

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_3y \\ \phi_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} \dfrac{-wL}{2} \\ \dfrac{-wL^2}{12} \\ \dfrac{-wL}{2} + 0 \\ \dfrac{wL^2}{12} + 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\Rightarrow \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_3y \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \\ 0 \\ 0 \end{Bmatrix} \quad \ldots (5)$$

Applying boundary conditions (Refer Fig. (ii)

(i) A is fixed. So, displacement d_{1y} and rotation ϕ_1 are zero.

(ii) At B, displacement $d_{2y} = 0$

(iii) C is fixed. So, displacement d_{3y} and rotation $\phi_3 = 0$

Substitute the above values in equation (5)

$$\Rightarrow \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation $d_{1y} = \phi_1 = d_{2y} = d_3y = \phi_3 = 0$.

So, delete first row first column, second row second column, third row third column, fifth row fifth column and sixth row sixth column of [K] matrix. Hence the equation reduces to

$$\frac{EI}{L^3}[8L^2]\phi_2 = \frac{wL^2}{12}$$

$$\Rightarrow \qquad \phi_2 = \frac{wL^3}{96\,EI}$$

$$\text{Slope or rotation}, \phi_2 = \frac{wL^3}{96\,EI}$$

$$\text{Result: Slope or rotation at B}, \phi_2 = \frac{wL^3}{96\,EI}$$

Example 2.26

For the beam and loading shown in fig (i), calculate the rotations at B and C.

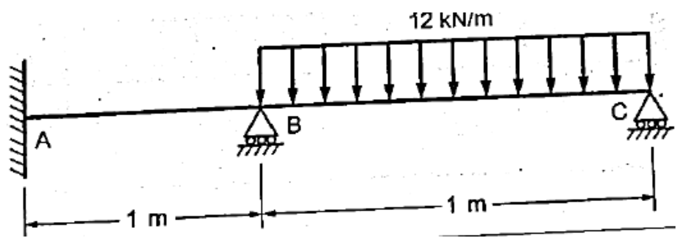


Fig. (i)

E = 210 GPa; I = 6×10^6 mm^4

Given:

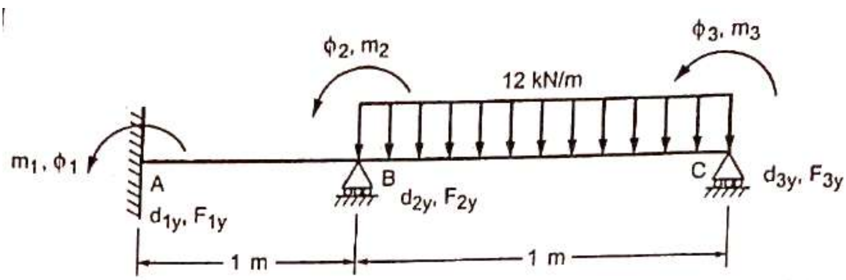


Fig. (ii)

Young's modulus, E = 210 GPa

= 210×10^9 Pa

E = 210×10^9 N/m^2

Moment of inertia, I = 6×10^6 mm^4

I = 6×10^{-6} m^4

For element 1, Length, L = 1 m

For element 2, Length, L = 1 m

Uniformly distributed load, w = 12 kN/m

$$w = 12 \times 10^3 \text{ N/m}$$

To find:

1. Slope or rotation at B, ϕ_2
2. Slope or rotation at C, ϕ_3

Solution: we can divide the beam into two elements as shown in Fig. (iii).

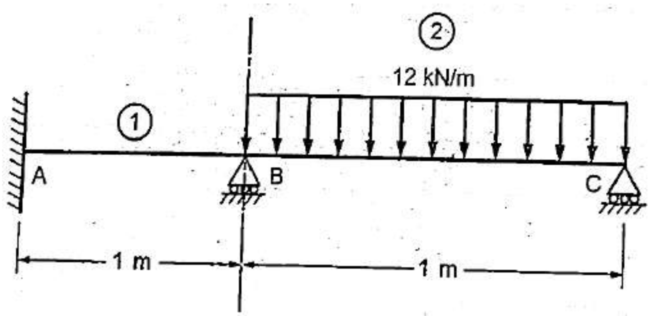


Fig. (iii)

For element (1) (Nodes 1, 2, 3, 4 i.e., $d_{1y}, \phi_1, d_{2y}, \phi_2$)

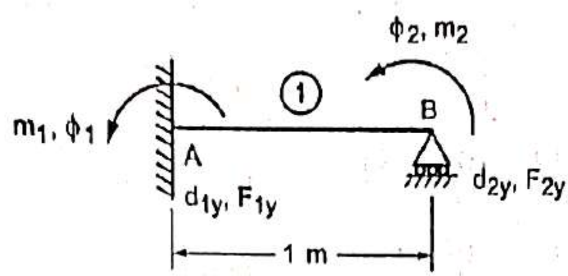


Fig.(iv)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} \quad \text{... (1)}$$

There is no load and moment on element (2). So, $F_{1y} = F_{2y} = 0$ and $m_1 = m_2 = 0$.

$$(1) \Rightarrow \frac{210\times10^{9}\times6\times10^{-6}}{(1)^{3}}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6L & 2 & -6 & 4^{2} \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \ldots(2) \quad [\because L = 1m]$$

For uniformly distributed load,

$$F_{2y} = \frac{-wL}{2}; \quad m_2 = \frac{-wL^2}{12}; \; F_{3y} = \frac{-wL}{2}; \quad m_3 = \frac{wL^2}{12} \quad \text{[Refer Table 2.1]}$$

$$\Rightarrow F_{2y} = \frac{-12\times10^{3}\times1}{2} = -6000N \qquad [\because w = 12\times10^{3}N/m; L = 1m]$$

$$\Rightarrow m_2 = \frac{-12\times10\text{^}3\times(1)^2}{12} = -1000N-m]$$

$$\Rightarrow F_{3y} = \frac{-12\times10^{3}\times1}{12} = -6000N-m]$$

$$\Rightarrow m_3 = \frac{12\times10^{3}\times(1)^2}{12} = 1000N-m]$$

$$(3) \Rightarrow \frac{210\times10^{9}\times6\times10^{-6}}{(1)^{3}}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6L & 2 & -6 & 4^{2} \end{bmatrix} \end{array}\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix}\begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \qquad \ldots(4)$$

Assemble the finite elements i.e., assemble the finite element equations (2) and (4)

$$\frac{210\times10^{9}\times6\times10^{-6}}{(1)^3}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 12+12 & -6+6 & -12 & 6 \\ 6 & 2 & -6+6 & 4+4 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_3y \\ \phi_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0-6000 \\ 0-1000 \\ -6000 \\ 1000 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$1.26\times10^{6}\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_3y \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix} \qquad \ldots(5)$$

Applying boundary conditions (Refer Fig. (ii)

(i) A is fixed. So, displacement d_{1y} and rotation ϕ_1 are zero.

(ii) At B, displacement $d_{2y} = 0$

(iii) At C, vertical displacement $d_{3y} = 0$

Substitute the above values in equation (5)

$$\Rightarrow \frac{EI}{L^3}\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \phi_2 \\ 0 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix}$$

In the above equation $d_{1y} = \phi_1 = d_{2y} = d_3y = 0$.

So, delete first row first column, second row second column, third row third column, fifth row fifth column of [K] matrix. Hence the equation reduces to

$$1.26 \times 10^6 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix}$$

Matrix multiplication,

$$\Rightarrow 1.26 \times 10^6 [8\phi_2 + 2\phi_3] = -1000 \quad \text{... (6)}$$

$$\Rightarrow 1.26 \times 10^6 [2\phi_2 + 4\phi_3] = 1000 \quad \text{... (7)}$$

Equation (7) × (-4) ⇒

$$\Rightarrow 1.26 \times 10^6 [-8\phi_2 - 16\phi_3] = -4000 \quad \text{... (8)}$$

Equation (6)) ⇒ $1.26 \times 10^6 [14\phi_3] = -1000$

Solving, $1.26 \times 10^6 [-14\phi_3] = -1000$

$$\Rightarrow \phi_3 = 2.834 \times 10^{-4} \text{ rad}$$

Substitute ϕ_3 value in equation (6).

$$\Rightarrow 1.26 \times 10^6 [-8\phi_2 + 2(2.834 \times 10^{-4})] = -1000$$

$$\Rightarrow \qquad \phi_2 = -1.70 \times 10^{-4} \text{ rad}$$

Result: (i) Slope or rotation at B, $\phi_2 = -1.70 \times 10^{-4}$ rad

(ii) Slope or rotation at C, $\phi_3 = 2.834 \times 10^{-4}$ rad

Example 2.27

Find the deflection at the point load and the slopes at the ends for the steel shaft which is simply supported at the bearing A and B as shown in Fig. (i).

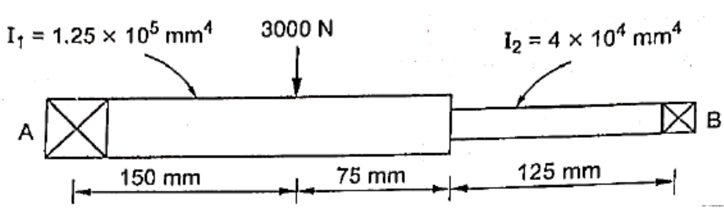


Fig (i)

Take E = 200 GPa.

Given:

Young's modulus, E = 200 GPa

$= 200 \times 10^9$ Pa (or) 200×10^9 N/m^2

$$E = 200 \times 10^9 \text{ N/m}^2$$
$$= 2 \times 10^5 \text{ N/mm}^2$$

Moment of inertia,

Element 1 and 2, $I_1 = 1.25 \times 10^5 \text{ mm}^4$

Element 3, $I_2 = 1.25 \times 10^5 \text{ mm}^4$

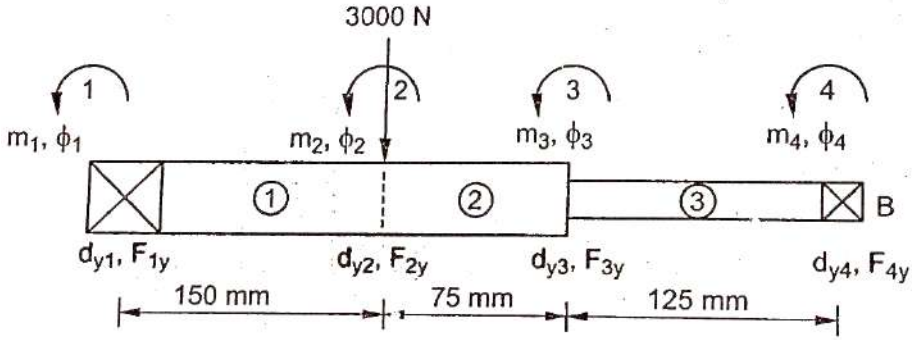


Fig (ii)

For element 1, Length, L_1 = 150 mm

For element 2, Length, L_2 = 75 m

For element 3, Length, L_3 = 125 m

Point load at node 2, F_2 = 3000 N

To find:

i) Deflection at the point load
ii) Slope at the ends

Solution: we can divide the beam into three elements as shown in Fig. (iii).

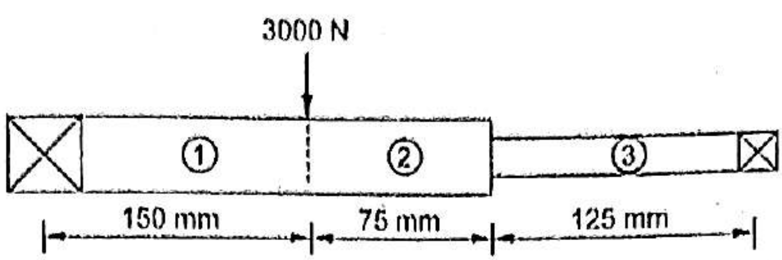


Fig. (iii)

For element (1) (Nodes 1, 2, 3, 4 i.e., $d_{1y}, \phi_1, d_{2y}, \phi_2$)

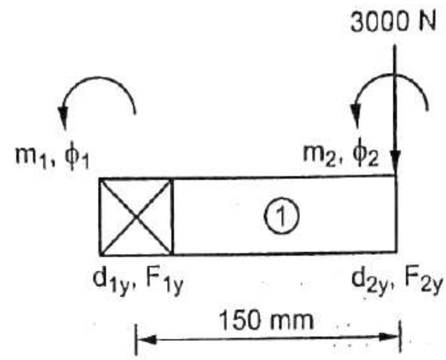


Fig.(iv)

Finite element equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} \qquad ...(1)$$

There is no load and moment on element (1).

So, $F_{1y} = F_{1y} = 0, m_1 = 0, F_{2y} = -3000\ N, m_2 = 0.$

$$\Rightarrow \frac{2\times 10^5 \times 1.25 \times 10^5}{150^3} \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 12 & 900 & -12 & 900 \\ 900 & 9\times 10^4 & 900 & 4.5\times 10^4 \\ -12 & -900 & 12 & -900 \\ 900 & 4.5\times 10^4 & -900 & 9\times 10^4 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{1y} \\ 0 \\ -3000 \\ 0 \end{Bmatrix} \qquad ...(2)$$

$$10^4 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 8.88 & 666 & -8.88 & 666 \\ 666 & 66600 & -666 & 33300 \\ -8.88 & -666 & 8.88 & -666 \\ 666 & 33300 & -666 & 66600 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -3000 \\ 0 \end{Bmatrix} \qquad ...(2)$$

Similarly,

For element (2) (Nodes 3, 4, 5, 6 i.e., $d_{2y}, \phi_2, d_{3y}, \phi_3$)

$$\Rightarrow \frac{2\times 10^5 \times 1.25 \times 10^5}{150^3}\begin{bmatrix} 12 & 450 & -12 & 450 \\ 450 & 22500 & -450 & 11250 \\ -12 & -450 & 12 & -450 \\ 450 & 11250 & -4500 & 22500 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix}$$

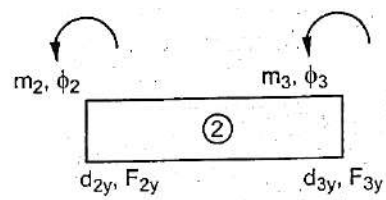


Fig. (v)

There is no load and moment on element (1).

So, $F_{2y} = 0, m_2 = 0, F_{3y} = 0, m_3 = 0$.

$$10^4\begin{bmatrix} 12 & 450 & -12 & 450 \\ 450 & 22500 & -450 & 11250 \\ -12 & -450 & 12 & -450 \\ 450 & 11250 & -4500 & 22500 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} F_{2y} \\ m_2 \\ F_{3y} \\ m_3 \end{Bmatrix} \qquad \text{... (4)}$$

$$\Rightarrow 10^4 \begin{array}{c} \begin{array}{cccc} \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} 71.16 & 2668.5 & -71.16 & 2668.5 \\ 2668.5 & 133425 & -2668.5 & 66712.5 \\ -71.16 & -2668.5 & 71.16 & -2668.5 \\ 2668.5 & 66712.5 & -2668.5 & 133425 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{array} \begin{Bmatrix} d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{... (4)}$$

Similarly,

Element (3) (Nodes 5, 6, 8, i.e., $d_{3y}, \phi_3, d_{3y}, \phi_4$)

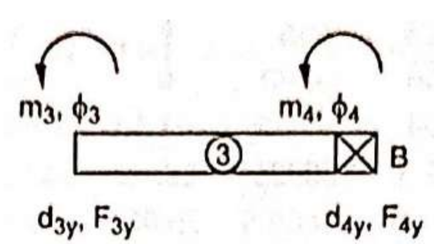


Fig. (vi)

$$\Rightarrow \frac{2\times 10^5\times 4\times 10^4}{125^3}\begin{bmatrix}12 & 750 & -12 & 750\\ 750 & 62500 & -750 & 31250\\ -12 & -750 & 12 & -750\\ 750 & 31250 & -750 & 62500\end{bmatrix}\begin{Bmatrix}d_{1y}\\ \phi_2\\ d_{3y}\\ \phi_3\end{Bmatrix}=\begin{Bmatrix}F_{3y}\\ m_3\\ F_{4y}\\ m_4\end{Bmatrix}$$

There is no load and moment on element (3).

So, $F_{3y}=0, F_{3y}=F_{3y}, m_3=m_4 0.$

$$\Rightarrow 10^4\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8\end{array}\\ \begin{bmatrix}4.92 & 307.5 & -4.92 & 307.5\\ 307.5 & 25625 & -307.5 & 12812.5\\ -4.92 & -307.5 & 4.92 & -307.5\\ 307.5 & 12812.5 & -307.5 & 25625\end{bmatrix}\end{array}\begin{array}{c}\\ 5\\ 6\\ 7\\ 8\end{array}\begin{Bmatrix}d_{3y}\\ \phi_3\\ d_{4y}\\ \phi_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ F_{4y}\\ 0\end{Bmatrix}\qquad \ldots(5)$$

Assemble the finite elements, i.e., assemble the finite element equations (3), (4) and (5).

$$\Rightarrow 10^4\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\end{array}\\ \begin{bmatrix}8.88 & 666 & -8.88 & 666 & 0 & 0 & 0 & 0\\ 666 & 66600 & -666 & 33300 & 0 & 0 & 0 & 0\\ -8.88 & -666 & 80.04 & 2002.5 & -71.61 & 2668.5 & 0 & 0\\ 666 & 33300 & 2002.5 & 200025 & -2668.5 & 66712.5 & 0 & 0\\ 0 & 0 & -71.16 & -2668.5 & 76.08 & -2361 & -4.92 & 307.5\\ 0 & 0 & 2668.5 & 66712.5 & -2361 & 159050 & -307.5 & 12812.5\\ 0 & 0 & 0 & 0 & -4.92 & -307.5 & 4.92 & -307.5\\ 0 & 0 & 0 & 0 & 3075 & 12812.5 & -307.5 & 25625\end{bmatrix}\end{array}\begin{array}{c}\\ 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\end{array}$$

$$\times\begin{Bmatrix}d_{1y}\\ \emptyset_1\\ d_{2y}\\ \emptyset_2\\ d_{3y}\\ \emptyset_3\\ d_{4y}\\ \emptyset_4\end{Bmatrix}=\begin{Bmatrix}F_{1y}\\ 0\\ -3000\\ 0\\ 0\\ 0\\ F_{4y}\\ 0\end{Bmatrix}\qquad \ldots(6)$$

Applying boundary conditions,

$d_{1y}=0,\quad \emptyset_1=\emptyset_1,\quad d_{2y}=d_{2y},\quad \emptyset_2=\emptyset_2$

$d_{3y}=d_{3y},\quad \emptyset_3=\emptyset_3,\quad d_{4y}=0,\quad \emptyset_4=\emptyset_4$

Since, $d_{1y} = 0, d_{4y} = 0$,neglect first row column, and seventh row seventh column in global stiffness matrix(6). Hence the equation reduces to

$$\Rightarrow 10^4 \begin{bmatrix} 66600 & -666 & 33300 & 0 & 0 & 0 \\ -666 & 80.04 & 2002.5 & -71.16 & 2668.5 & 0 \\ 33300 & 2002.5 & 200025 & -2668.5 & 66712.5 & 0 \\ 0 & -71.16 & -2668.5 & 76.08 & -2361 & 307.5 \\ 0 & 2668.5 & 66712.5 & -2361 & 159050 & 12812.5 \\ 0 & 0 & 0 & 307.5 & 12812.5 & 25625 \end{bmatrix} \begin{Bmatrix} \emptyset_1 \\ d_{2y} \\ \emptyset_2 \\ d_{3y} \\ \emptyset_3 \\ \emptyset_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -3000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{... (7)}$$

From equation (7)

$$10^4\left[66600\,\emptyset_1 - 666\,d_{2y} + 33300\,\emptyset_2\right] = 0 \quad \text{.... (8)}$$

$$10^4\left[-666\,\emptyset_1 + 80.04\,d_{2y} + 2002.5\,\emptyset_2 - 71.16\,d_{3y} + 2668.5\,\emptyset_3\right] = -3000 \quad \text{... (9)}$$

$$10^4\left[33300\,\emptyset_1 + 2002.5\,d_{2y} + 20025\,\emptyset_2 - 2668.5\,d_{3y} + 66712.5\,\emptyset_3\right] = -3000 \quad \text{... (10)}$$

$$10^4\left[-71.16\ d_{2y} - 2668.5\ \emptyset_2 + 76.08\,d_{3y} - 2361\ \emptyset_3 + 307.5\,\emptyset_4\right] = 0 \quad \text{... (11)}$$

$$10^4\left[2668.5\ d_{2y} + 66712.5\ \emptyset_2 - 2361\,d_{3y} + 159050\ \emptyset_3 + 12812.5\,\emptyset_4\right] = 0 \quad \text{... (12)}$$

$$10^4\left[307.5\ d_{3y} + 12812.5\ \emptyset_3 + 25625\,\emptyset_4\right] = 0 \quad \text{... (13)}$$

Equaiton (8) $\Rightarrow\ \emptyset_2 = \dfrac{-66600\,\emptyset_1 + 666\,d_2y}{33300}$

$$\emptyset_2 = -2\,\emptyset_1 + 0.02\,d_{2y}$$

$Equaiton\ (13)\ \Rightarrow\ \emptyset_3 = \dfrac{-307.5\,d_{3y} - 25625\,\emptyset_4}{12812.5}$

$$\emptyset_3 = -0.024\,d_{3y} - 2\ \emptyset_4$$

Substituting the values of $\emptyset_2, \emptyset_3$ in equation (9), (10), (11), (12), we get

From equation (9) ⇒

$$10^4\left[-666\,\emptyset_1 + 80.04\,d_{2y} + 2002.5\ \ (-2\,\emptyset_1 + 0.02\,d_{2y}) - 71.16\,d_{3y} + 2668.5(-0.024\,d_{3y} - 2\,\emptyset_4\right] = -3000$$

$$\Rightarrow\ \ -666\,\emptyset_1 + 80.04\,d_{2y} - 4005\,\emptyset_1 + 40.05\,d_{2y} - 71.16\,d_{3y} - 64.044\,d_{3y} - 5337\,\emptyset_4 = -0.3$$

$$\Rightarrow -4671\,\emptyset_1 + 120.1\,d_{2y} - 135.2\,d_{3y} - 5337\,\emptyset_4 = -0.3 \ldots (14)$$

We know that,

From equation (10) ⇒

$$10^4\left[33300\,\emptyset_1 + 2002.5\,d_{2y} + 200025(-2\,\emptyset_1 + 0.02\,d_{2y}) - 2668.5\,d_{3y} + 6671.5\,(-0.024\,d_{3y} - 2\,\emptyset_4\right] = 0$$

$$\Rightarrow\ \ -366750\,\emptyset_1 + 6003\,d_{2y} - 4269.6\,d_{3y} - 133425\,\emptyset_4 = 0 \ldots (15)$$

From equation (11) ⇒

$$10^4\left[-71.16\,d_{2y} - 2668.5\,(-2\,\emptyset_1 + 0.02\,d_{2y}) + 76.08\,d_{3y} - 2361\,(-0.024\,d_{3y} - 2\,\emptyset_4) + 307.5\,d_{4y}\right] = 0$$

$$\Rightarrow\ \ 5337\,\emptyset_1 - 124.5\,d_{2y} + 132.7\,d_{3y} + 5029.5\,d_{4y} = 0 \qquad \ldots (16)$$

From equation (12) ⇒

$$10^4\left[2668.5\,d_{2y} + 66712.5(-2\,\emptyset_1 + 0.02\,d_{2y}) - 2361\,d_{3y} - 159050\,(-0.024\,d_{3y} - 2\,\emptyset_4) + 12812.5\,\emptyset_4\right] = 0$$

$$\Rightarrow\ \ -133425\,\emptyset_1 + 4002.8\,d_{2y} - 6178.2\,d_{3y} - 305288\,\emptyset_4 = 0 \qquad \ldots (17)$$

From equation (17) ⇒

$$d_{3y} = \frac{133425\,\emptyset_1 + 4002.8\,d_{2y} + 305288\,\emptyset_4}{-6178.2}$$

$$d_{3y} = -21.6\,\emptyset_1 + 0.65\,d_{2y} - 49.4\,\emptyset_4 \qquad \ldots (18)$$

Substitute the equation (18) in equations (14), (15) and (16),

From equation (14) ⇒

$$-4671\ \emptyset_1 + 120.1\ d_{2y} - 135.2(-21.6\ \emptyset_1 + 0.65\ d_{2y} - 49.4\ \emptyset_4) - 5337\ \emptyset_4 = -0.3$$

$$\Rightarrow \quad -1750.7\ \emptyset_1 + 32.28\ d_{2y} + 1341.9\ \emptyset_4 = -0.3 \qquad \text{... (19)}$$

From equation (15) ⇒

$$-366750\ \emptyset_1 + 6003\ d_{2y} - 4269.6(-21.6\ \emptyset_1 + 0.65\ d_{2y} - 49.4\ \emptyset_4) - 133425\ \emptyset_4 = 0$$

$$\Rightarrow \quad -274527\ \emptyset_1 + 3355.8\ d_{2y} + 77493\ \emptyset_4 = 0 \qquad \text{... (20)}$$

From equation (16) ⇒

$$-5337\ \emptyset_1 - 124.5\ d_{2y} - 132.7(-21.6\ \emptyset_1 + 0.65\ d_{2y} - 49.4\ \emptyset_4) + 5029.5\ \emptyset_4 = 0$$

$$\Rightarrow \quad 2470.7\ \emptyset_1 - 38.2\ d_{2y} - 1525.9\ \emptyset_4 = 0 \qquad \text{... (21)}$$

$$\Rightarrow \emptyset_4 = \frac{2470.7\ \emptyset_1 - 38.2\ d_{2y}}{1525.9} \qquad \text{... (22)}$$

$$\Rightarrow \emptyset_4 = 1.62\ \emptyset_1 - 0.025\ d_{2y} \qquad \text{... (23)}$$

Substitute the equation (23) in equations (20), we get

Form equation (20) ⇒

$$-274527\ \emptyset_1 + 3355.8\ d_{2y} + 77493\ (1.62\ \emptyset_1 - 0.025\ d_{2y}) = 0$$

$$\Rightarrow \quad -148988\ \emptyset_1 + 1418.5\ d_{2y} = 0$$

$$\therefore \Rightarrow \quad -148988\ \emptyset_1 = -1418.5\ d_{2y}$$

$$d_{2y} = 105\ \emptyset_1 \qquad \text{... (24)}$$

Substitute equation (240 and (25) in equation (19),

Equation (19) ⇒

$$-1750.7\, \emptyset_1 + 32.2\,(105\, \emptyset_1) + 1341.6\,[1.62\, \emptyset_1 - 0.025\,(105\, \emptyset_1)] = -0.3$$

$$281.7\, \emptyset_1 = -0.3$$

$$\emptyset_1 = -0.0011\, rad$$

∴ Subsittue the value $(\emptyset_1)$ in equaiton (24),

$$d_{2y} = 105(-0.0011)$$

$$d_{2y} = -0.112\, mm$$

∴ $Subsittue\ the\ value\ (\emptyset_1)\ and\ (d_{2y}) in\ equaiton\ (23),$

$$\emptyset_4 = 1.62(-0.0011) - 0.025(-0.112)$$

$$\emptyset_4 = 0.0010\, rad$$

Result:

1. Deflection at the loaded point, $d_{2y} = -0.112\, mm$
2. Slope at the left bearing, $\emptyset_1 = -0.0011\, rad$
3. Slope at the right bearing $\emptyset_4 = 0.0010\, rad$

2.13. QUADRATIC BAR ELEMENT

2.13.1. Derivation of Shape Functions for One-Dimensional Quadratic Bar Element

Consider a quadratic bar element with nodes 1,2 and 3 as shown in Fig.(i). u_1, u_2 and u_3 are the displacements at the respective nodes. So, u1, u2 and u3 are considered as degrees of freedom of this quadratic bar element.

Since the element has got three nodal displacements, it will have three generalized coordinates.

$$u = a_0 + a_1 x + a_2 x^2 \qquad \text{... (1)}$$

Where, $a - 0\ , a_1, a_2$ are global or generalized coordinates. Writing the equation (2.47) in matrix form,

$$u = [\,1\; x\; x^2]\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} \qquad \ldots (2)$$

At node 1, $u = u_1$, $\quad x = 0$

At node 2, $u = u_2$, $\quad x = l$

At node 3, $u = u_3$, $\quad x = \frac{l}{2}$

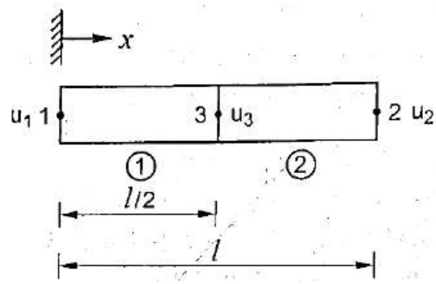


Fig. (i). Quadratic bar element

Substitute the above values in equation (2.47),

$$u_1 = a_0 \qquad \ldots (3)$$

$$u_2 = a_0 + a_1 l + a_2 l^2 \qquad \ldots (4)$$

$$u_3 = a_0 + a_1\left(\frac{l}{2}\right) + a_2\left(\frac{l}{2}\right)^2 \qquad \ldots (5)$$

Substitute the equation (3) in equation (4) and (5),

$$Equation\ (2.50) \Rightarrow u_2 = u_1 + a_1 l + a_2 l^2 \qquad \ldots (6)$$

$$Equation\ (2.51) \Rightarrow u_3 = u_1 + \frac{a_1 l}{2} + \frac{a_2 l^2}{4} \qquad \ldots (7)$$

$$Equation\ (2.52) \Rightarrow u_2 - u_1 = a_1 l + a_2 l^2 \qquad \ldots (8)$$

$$Equation\ (2.53) \Rightarrow u_3 - u_1 = \frac{a_1 l}{2} + \frac{a_2 l^2}{4} \qquad \ldots (9)$$

Arranging the equation (2.54) and (2.55) in matrix form,

$$\begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix}^{-1} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix}$$

$$= \frac{1}{(\frac{l^3}{4} - \frac{l^3}{2})} \begin{bmatrix} \frac{l^2}{4} & l^2 \\ \frac{-l}{2} & l \end{bmatrix} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix}$$

$$\left[Note: \because \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}\, a_{22} - a_{12}\, a_{21})} \times \begin{bmatrix} a_{22} - a_{12} \\ -a_{21}\; a_{11} \end{bmatrix} \right]$$

$$\Rightarrow \qquad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{\left(\frac{-l^3}{4}\right)} \begin{bmatrix} \frac{l^2}{4} & -l^2 \\ \frac{-l}{2} & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix} \qquad \dots (10)$$

$$\Rightarrow \qquad a_1 = \frac{-4}{l^3} \left[\frac{l^2}{4} (u_2 - u_1) - l^2 (u_3 - u_1) \right] \qquad \dots (11)$$

$$\Rightarrow \qquad a_2 = \frac{-4}{l^3} \left[\frac{-l}{4} (u_2 - u_1) - l (u_3 - u_1) \right] \qquad \dots (12)$$

$Equation\ (2.57) \Rightarrow$

$$a_1 = \frac{-4}{l^3} \left[\frac{l^2 u_2}{4} - \frac{l^2 u_1}{4} - l^2 u_3 + l^2 u_1 \right]$$

$$= \frac{-4\, l^2 u_2}{4\, l^3} + \frac{4\, l^2 u_1}{4\, l^3} + \frac{4\, l^2 u_3}{l^3} - \frac{4\, l^2 u_1}{l^3}$$

$$= \frac{-\, u_2}{l} + \frac{u_1}{l} + \frac{4\, u_3}{l} - \frac{4\, u_1}{l}$$

$$a_1 = \frac{-3\, u_1}{l} - \frac{u_2}{l} + \frac{4\, u_3}{l} \qquad \dots (13)$$

Equation (2.58) $\Rightarrow$

$$a_2 = \frac{-4}{l^3} \left[\frac{-l\, u_2}{4} + \frac{l}{2}\, u_1 + l\, u_3 - l\, u_1 \right]$$

$$= \frac{4\,l\,u_2}{2\,l^3} - \frac{4\,l}{2\,l^3}u_1 - \frac{4\,l}{l^3}u_3 - \frac{4\,l}{l^3}\,u_1$$

$$= \frac{2\,u_2}{l^2} - \frac{2}{l^2}u_1\frac{4}{l^2}u_3 - \frac{4}{l^2}u_1$$

$$a_2 = \frac{2}{l^2}u_1 - \frac{2\,u_2}{l^2} + \frac{4}{l^2}\,u_3 \qquad \text{... (14)}$$

Arranging the equation (2.49), (2.59) and (2.60) in matrix form,

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{... (15)}$$

Substitute the equation (2.61) in equation (2.48),

$$\{u\} = [\,1 \quad x \quad x^2] \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{... (16)}$$

$$\{u\} = \left[\left(1 - \frac{3}{l}\,x + \frac{2\,x^2}{l^2}\right)\left(\frac{-x}{l} + \frac{2\,x^2}{l^2}\right)\left(\frac{4\,x}{l} + \frac{4\,x^2}{l^2}\right)\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = [N_1\;N_2\;N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = N_1 u_1 + N_2 u_2\, N_3 u_3 \qquad \text{... (17)}$$

Where, shape functions,

$$N_1 = 1 - \frac{3\,x}{l} + \frac{2\,x^2}{l^2}$$

$$N_2 = \frac{-x}{l} + \frac{2\,x^2}{l^2}$$

$$N_3 = \frac{4\,x}{l} - \frac{4\,x^2}{l^2}$$

2.13.2. Solved Problems on Quadratic Bar Element

Example 2.28

A Steel bar of length 800mm is subjected to an axial load of 3kN as shown in fig (i). Find the nodal displacement of the bar and load vectors,

Take E=2*10^5N/mm^2, A=300mm^2, and mass density ρ =7800kg/m^3

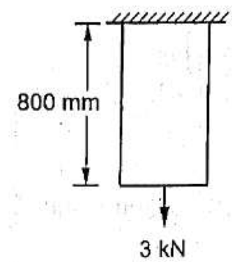


Fig. (i).

Given: Length, l = 800 mm

Load, F = 3 kN = 3×10^3 N

Young's modulus, E = 2 × 105 N/mm^2

Area, A = 300 mm^2

Mass desnity, ρ = 7800 kg/m^2 – 76518 N/m^3 = 7.6518×10^{-5} N/mm^3

To find: (i) Nodal displacements

(ii) Load vectors

Solution: we can divide the bar into two elements as shown in Fig. (ii).

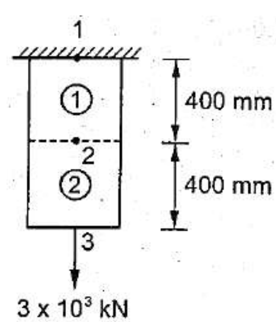


Fig. (ii)

The bar is subjected to self-weight. So, we have to find the body force acting at nodal points 1, 2 and 3.

We know that,

Force vector for quadratic bar element,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \rho\, A\, l \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix}$$

$$= 7.6518 \times 10^{-5} \times 300 \times 800 \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix}$$

$$= 18.364 \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 3.06 \\ 3.06 \\ 12.24 \end{Bmatrix} \qquad ..(1)$$

At point load of 3×10^3 N is acting at node 3 as shown in Fig. (ii). So, add 3×10^3 N in F_3 vector,

$$\Rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 3.06 \\ 3.06 \\ 12.24 + 3 \times 10^3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{Bmatrix} \qquad ...(2)$$

$\therefore$ Finite element equation for one dimensional quadratic bar element is given by,

$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{E\ A}{3\ l} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (3)$$

Substitute the equation (2) in equation (3),

$$\begin{Bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{Bmatrix} = \frac{2 \times 10^5 \times 300}{3\ \times 800} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{Bmatrix} = 25000 \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (4)$$

Applying the boundary conditions, i.e.,

At node 1, displacement, $u_1 = 0$ (fixed)

At node 2, displacement, $u_2 = u_2$

At node 3, displacement, $u_3 = u_3$

Substituting u_1, u_2 *and* u_3 values in equation (4),

$$\Rightarrow \qquad \begin{Bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{Bmatrix} = 25000 \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

In the above equation $u_1 = 0,$ so, neglect first row and first column of [k] matrix. The reduced equation is,

$$= 25000 \begin{bmatrix} 7 & -8 \\ -8 & 16 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 3.06 \\ 3012.24 \end{Bmatrix}$$

$$\Rightarrow \qquad 25000\ [7u_2 - 8\ u_3] = 3.06 \qquad \ldots (5)$$

$$\Rightarrow 25000\ [-8\ u_2 + 16\ \ u_3] = 3012.24 \qquad \ldots (6)$$

$$\therefore\ Equaiton\ (6) \times 7 \Rightarrow\ 25000[\,-56\ u_2 + 112\ u_3] = 2.108\ \times 10^4$$

$$Equaiton\ (5) \times 8 \Rightarrow \qquad 25000[\ 56\ u_2 - 56\ u_3] = 24.48$$

$solving \qquad 25000\ (56\,u_3) = 2.1104\ \times 10^4$

$\Rightarrow \qquad u_3 = 0.0150\,mm$

Substitute u_3 value in equation (5),

$\Rightarrow \qquad 25000[\,7\,u_2 - 8(0.015)] = 3.06$

$\Rightarrow \qquad u_2 = 0.0171\,mm$

Result:

Load vector, $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{Bmatrix}$

Nodal displacement, $u_1 = 0$

$u_2 = 0.0171\,mm$

$u_3 = 0.0150\,mm$

UNIT 3

TWO DIMENSIONAL PROBLEMS

3.1. INTRODUCTION

This chapter considers the two dimensional finite element. Two dimensional elements are defined by three or more nodes in a two dimensional plane (i.e., x, y plane). The basic element useful for two dimensional analysis is the triangular element. The simplest two dimensional elements have corner nodes as shown in Fig.3.1. A quadrilateral (special forms of rectangle and parallelogram) element can be obtained by assembling two or four triangular elements, as shown in Fig.3.2. They are often used to model a wide range of engineering problems.

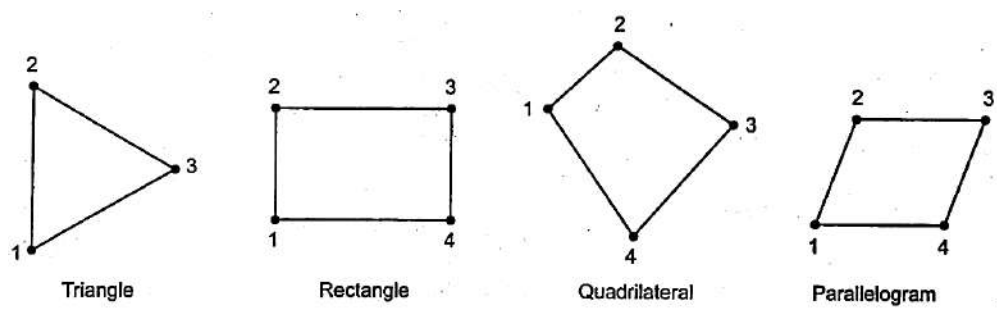


Fig. 3.1. Two dimensional elements

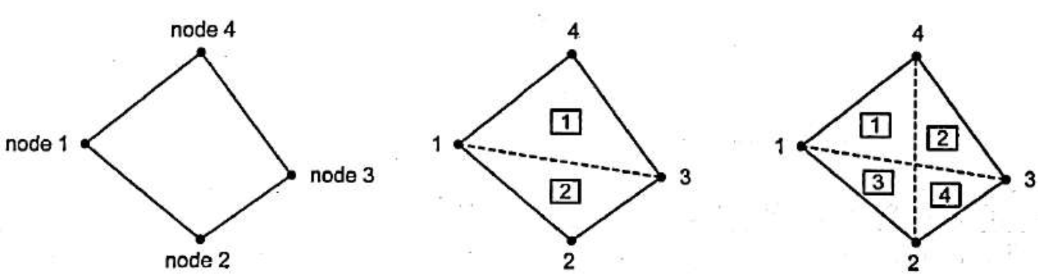


Fig. 3.2. A quadrilateral element as an assemblage of two or four triangular elements

The two dimensional analysis of hydraulic cylinder rod end with plane strain triangular elements is shown in Fig.3.3.

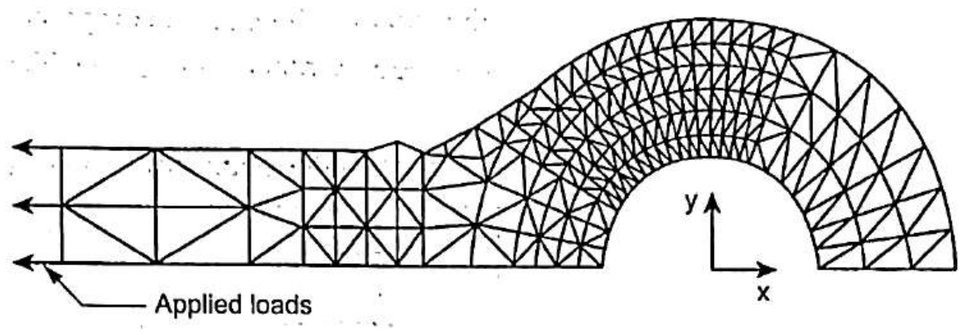


Fig. 3.2. Two dimensional analysis of hydraulic cylinder rod end

The two dimensional finite element formulation follows the same steps which is used in the one dimensional problems. The displacements and distributed body force values are functions of the position indicated by (x, y).

The displacement vector u is given by, $u = \begin{Bmatrix} u \\ v \end{Bmatrix}$

Where u and v are the x and y components of u respectively.

The stresses and strains are given as,

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$e = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

Where, σ → Normal stress

τ → Shear stress

e → Normal strain

γ → Shear strain.

Body force is given by, $F = \begin{Bmatrix} Fx \\ Fy \end{Bmatrix}$

3.2. PLANE STRESS AND PLANE STRAIN

The two dimensional element is extremely important for the following two analysis.

(i) Plane stress analysis.

(ii) Plane strain analysis.

(i) Plane Stress Analysis

Plane stress is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.

Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the x-y plane can be considered to be under plane stress.

Plates with holes and plates with fillets are coming under plane stress analysis problems.

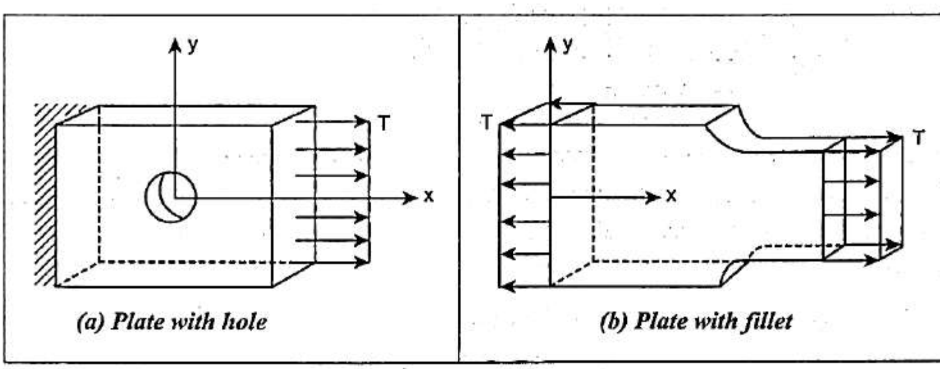


Fig. 3.4 Plane stress problems: (a) plate with hole; (b) plate with fillet

where, T→ Surface tractions (i.e., pressure acting on the surface edge or face of a member, unit→ Force/Area → N/m^2)

Normal stress, $\sigma_z = 0$

Shear stress τ_{xz} and $\tau_{yz} = 0$

(ii) Plane strain analysis

Plane strain is defined to be a state of strain in which the strain normal to the xy plane and the shear strains are assumed to be zero.

Dams and pipes subjected to loads that remain constant over their lengths are coming under plane strain analysis problems.

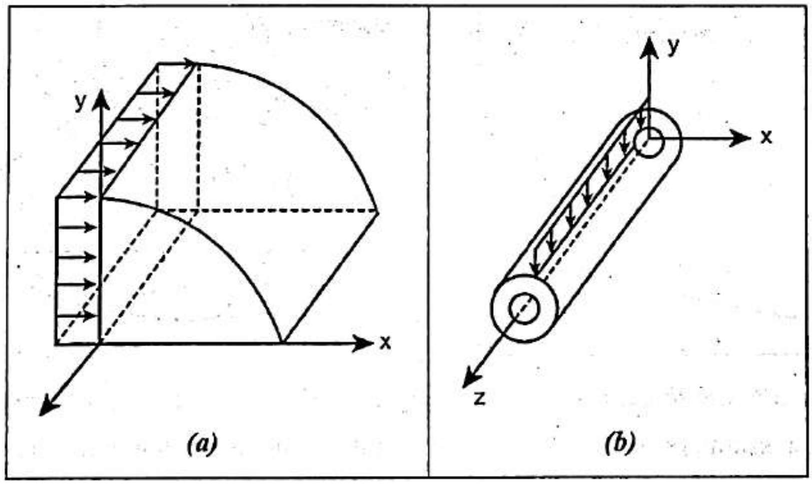


Fig. 3.5 Plane strain problems; (a) dam subjected to horizontal loading; (b) pipe subjected to a vertical load

Here, Normal stress, $e_z = 0$

Shear stresses, γ_{xz} and $\gamma_{yz} = 0$

3.3. FINITE ELEMENT MODELLING

Finite element modelling consists of the following:

(i) Discretization of structure.

(ii) Numbering of nodes.

(i) Discretization

The art of subdividing a structure into a convenient number of smaller components is known as discretization. In two dimensional problems, three kinds of finite elements are used.

They are:

(i) Triangular element.

(ii) Rectangular element.

(iii) Quadrilateral element.

In truss, the above three elements are physically present. But in a continuum, the above three elements exist only in our imagination. The continuum shown in Fig.3.6 is discretized into eight triangular element as shown in Fig.3.7. The points where the corners of the triangles meet are called nodes. Each triangle formed by three nodes and three sides is called an element.

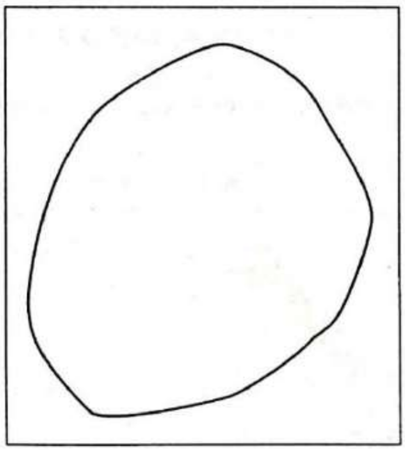


Fig. 3.6. Continuum

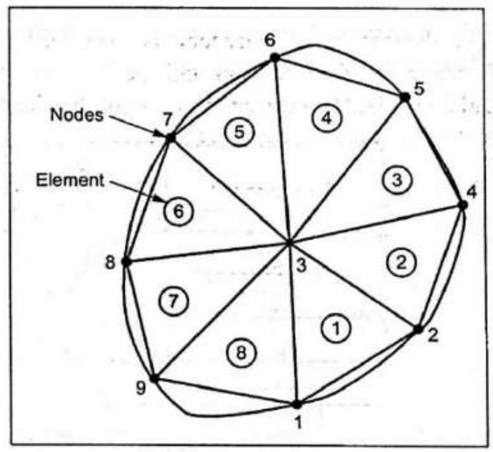


Fig. 3.7. Discretized into eight triangular elements

The element numbers are circled to distinguish from node numbers. The cross-section area, traction force and body force are constant within each element. But these are differ in magnitude from element to element. Better results are obtained by increasing the number of elements.

In Fig.3.7, the triangular elements fill the entire region except a small region at the boundary. This unfilled region can be eliminated by choosing smaller elements or elements with curved boundaries.

(ii) Numbering of Nodes

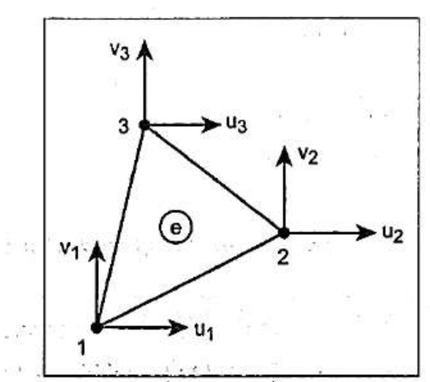


Fig. 3.8. Triangular element

In one dimensional problem, each node is allowed to move only in ±x direction. But in two dimensional problem, each node is permitted to move in the two directions i.e., x and y. Hence each node has two degrees of freedom (Nodal displacements). A three node finite element model is shown in Fig.3.8 has six degrees of freedom.

The element connectivity table is given for Fig.3.7. The heading 1 and 2 refer to the local node numbers of an element and the corresponding node numbers on the body are called global numbers. Connectivity thus establishes the local-global correspondence.

3.4. CONSTANT STRAIN TRIANGULAR (CST) ELEMENT

A three noded triangular element is known as constant strain triangular (CST) element which is shown in Fig.3.9. It has six unknown displacement degrees of freedom (u_1 v_1, u_2 v_2, u_3 v_3). The element is called CST because it has a constant strain throughout it.

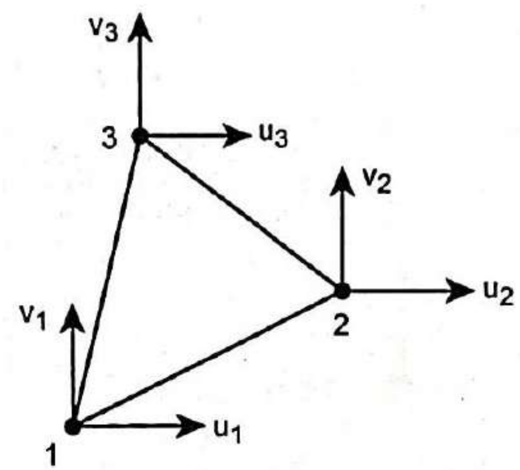


Fig. 3.9. Constant strain triangular element

3.5. SHAPE FUNCTION DERIVATION FOR THE CONSTANT STRAIN TRIANGULAR ELEMENT (CST)

We begin this section with the development of the shape function for a basic two dimensional finite element, called constant strain triangular element (CST).

We consider this CST element because its derivation is the simplest among the available two dimensional elements.

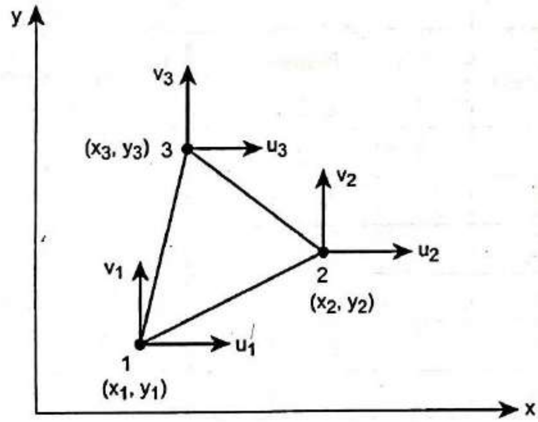


Fig. 3.10. Three noded CST element

Consider a typical CST element with nodes 1, 2 and 3 as shown in Fig.3.10 Let the nodal displacements be u_1 , v_1, u_2 , v_2, u_3 and v_3

Displacement $$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

since the CST element has got two degrees of freedom at each node (*u, v*) the total degrees of freedom is 6. Hence it has 6 generalized coordinates,

Let, $$u = a_1 + a_2 x + a_3 y \qquad \ldots(3.1)$$

$$v = a_4 + a_5 x + a_6 y \qquad \ldots(3.2)$$

Where, a_1, a_2, a_3, a_4, a_5 and a_6 are global or generalized co-ordinates.

$\Rightarrow$ $$u_1 = a_1 + a_2 x_1 + a_3 y_1$$

$$u_2 = a_1 + a_2 x_2 + a_3 y_2$$

$$u_3 = a_1 + a_2 x_3 + a_3 y_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (3.3)$$

$$Let \qquad D = \begin{bmatrix} + & - & + \\ 1 & x_1 & y_1 \\ - & + & - \\ 1 & x_2 & y_2 \\ + & - & + \\ 1 & x_3 & y_2 \end{bmatrix}$$

We know, $\qquad D^{-1} = \dfrac{C^T}{|D|} \qquad \ldots (3.4)$

Find the co-factors of matrix D.

$$C_{11} = +\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2)$$

$$C_{12} = -\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = (y_3 - y_2) = y_2 - y_3$$

$$C_{13} = +\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = x_3 - y_2$$

$$C_{21} = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = (x_3 y_1 - x_1 y_3)$$

$$C_{22} = +\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$C_{23} = -\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$C_{31} = +\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$C_{32} = -\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} = (y_2 - y_1) = y_1 - y_2$$

$$C_{33} = +\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$\Rightarrow \qquad C = \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - y_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix}$$

$$\Rightarrow \quad C^T = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - y_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \quad \text{... (3.5)}$$

We know, $\quad D = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$

$$|D| = 1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2) \quad \text{... (3.6)}$$

Substitute C^T and D values in equation (3.4),

$$\Rightarrow \quad D^{-1} = \frac{1}{(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)} \times \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1)) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - y_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

Substitute D^{-1} values in equation (3.3),

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)} \times \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1)) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - y_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{... (3.7)}$$

The area of the triangle can be expressed as a function of the x, y co-ordinates of the nodes 1, 2 and 3.

$$\Rightarrow \quad A = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$|A| = \frac{1}{2}[1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)]$$

$$\Rightarrow \quad 2A = (x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2) \quad \text{... (3.8)}$$

Substitute 2A values in equation (3.7),

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \times \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1)) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - y_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{... (3.9)}$$

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \times \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{... (3.10)}$$

Where, $p_1 = x_2y_3 - x_3y_2;\quad p_2 = x_3y_1 - x_1y_3;\quad p_3 = x_1y_2 - x_2y_1$

$q_1 = y_2 - y_3;\quad q_2 = y_3 - y_1;\quad q_3 = y_1 - y_2$

$r_1 = x_3 - x_2;\quad r_2 = x_1 - x_3;\quad r_3 = x_2 - x_1$

From equation (3.1), we know that,

$$u = a_1 + a_2x + a_3y$$

We can write this equation in matrix form,

$$u = [1 \;\; x \;\; y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$ value, from equation no. (3.10)

$$\Rightarrow u = [1 \;\; x \;\; y] \times \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [1 \;\; x \;\; y] \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [p_1 + q_1x + r_1y \quad p_2 + q_2x + r_2y \quad p_3 + q_3x + r_3y] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[\because (1 \times \underbrace{3) \times (3} \times 3) = 1 \times 3$$

$$u = \left[\frac{p_1 + q_1x + r_1y}{2A} \quad \frac{p_2 + q_2x + r_2y}{2A} \quad \frac{p_3 + q_3x + r_3y}{2A}\right] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

$$u = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (3.11)$$

$$\text{Similarly,} \quad v = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad \ldots (3.12)$$

$$\text{Where, Shape Function,} \quad N_1 = \frac{p_1 + q_1x + r_1y}{2A}$$

$$N_2 = \frac{p_2 + q_2x + r_2y}{2A}$$

$$N_3 = \frac{p_3 + q_3x + r_3y}{2A}$$

Assembling the equations (3.11) and (3.12) in matrix form,

$$\text{Displacement function, u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \ldots (3.13)$$

3.6 STRAIN – DISPLACEMENT MATRIX [B] FOR CST ELEMENT

Displacement function for CST element is given by,

$$\text{u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

or we can write

$$\text{u} = \text{N}_1u_1 + N_2u_2 + N_3u_3$$

$$\mathrm{u} = \mathrm{N}_1 v_1 + N_2 v_2 + N_3 v_3$$

The strain components for CST element are,

$$\text{Normal strain, } \mathrm{e_x} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \mathrm{e_x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$$

$$\text{Normal strain, } \mathrm{e_y} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \mathrm{e}_y = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3$$

$$\text{Shear strain, } \gamma_{\mathrm{xy}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\Rightarrow \gamma_{\mathrm{xy}} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 + \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3$$

Arranging the strains $\mathrm{e_x}, e_y$ and γ_{xy} in matrix form,

$$\Rightarrow \begin{Bmatrix} e_x \\ e_y \\ \gamma_{\mathrm{xy}} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{... (3.14)}$$

From equation (3.11) or (3.12), we know that,

$$\text{Shape Function, } N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

Partial differentiation,

$$\frac{\partial N_1}{\partial x} = \frac{q_1}{2A}; \qquad \frac{\partial N_2}{\partial x} = \frac{q_2}{2A}; \qquad \frac{\partial N_3}{\partial x} = \frac{q_3}{2A}$$

$$\frac{\partial N_1}{\partial y} = \frac{r_1}{2A}; \qquad \frac{\partial N_2}{\partial y} = \frac{r_2}{2A}; \qquad \frac{\partial N_3}{\partial y} = \frac{r_3}{2A}$$

Substitute $\frac{\partial N_1}{\partial x}, \frac{\partial N_2}{\partial x}, \frac{\partial N_3}{\partial x}, \frac{\partial N_1}{\partial y}, \frac{\partial N_2}{\partial y}$ and $\frac{\partial N_3}{\partial y}$ values in equation (3.14),

$$(3.14) \Rightarrow \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

The above equation is in the form of { e } = [B] { u }

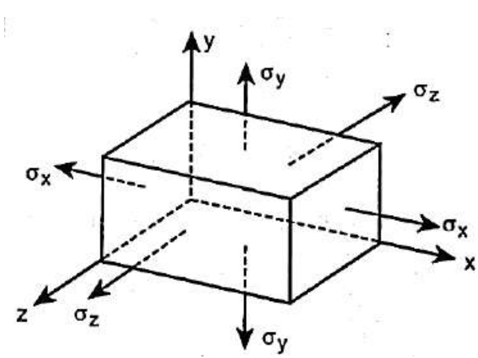
Where, [B] = Strain – Displacement matrix $= \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$ ... (3.15)

$$\begin{aligned} where,\ q_1 &= y_2 - y_3 \\ q_2 &= y_3 - y_1 \\ q_3 &= y_1 - y_2 \\ r_1 &= x_3 - x_2 \\ r_2 &= x_1 - x_3 \\ r_3 &= x_3 - x_1 \end{aligned}$$

[From equation no. (3.10)]

3.7. STRESS-STRAIN RELATIONSHIP MATRIX OR CONSTITUTTIVE MATRIX [D] FOR TWO DIMENSIONAL ELEMENT

Consider a three dimensional body which is subjected to the stresses σ_x, σ_y and σ_z independently as shown in Fig. 311.



Hooke's law states that when a material is loaded within its elastic limit, the stress is directly proportional to the strain.

i.e., stress $\propto$ strain

$$\sigma \propto e$$

$$\sigma = Ee$$

$$e = \frac{\sigma}{E}$$

Where, e = Strain

σ = Stress, N/mm^2

E = Yound's modulus or Modulus of elasticity, N/mm^2

The stress in the *x* direction produces a positive strain in *x* direction as shown in Fig. 3.12.

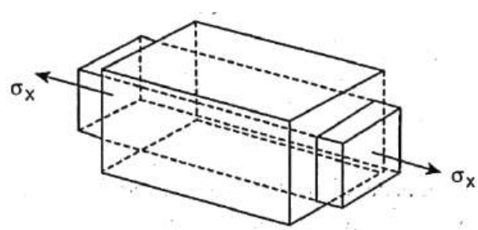


Fig. 3.12

Strain, $$e'_x = \frac{\sigma_x}{E} \quad \text{... (3.16)}$$

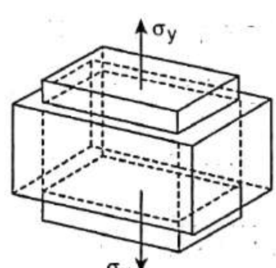


Fig. 3.13

Fig. 3.13 shows the positive stress in the *y* direction produces a negative strain in the *x* direction as a result of Poisson's effect which is given by,

$$-e'_x = \frac{v\sigma_x}{E}$$

$$\Rightarrow \quad e'_x = \frac{-v\sigma_x}{E} \qquad \text{... (3.17)}$$

Where, $v \rightarrow$ Poisson's ratio.

Similarly, the stress in the *z* direction produces a negative strain in the *x* direction as shown in Fig. 3.14.

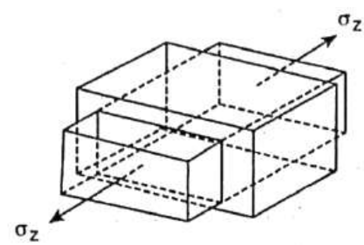


Fig. 3.14

$$-e'''_x = \frac{v\sigma_z}{E}$$

$$\Rightarrow \quad e'''_x = \frac{-v\sigma_z}{E} \qquad \text{... (3.18)}$$

By applying superposition principle to the equations (3.16), (3.17) and (3.18), we get

$$e_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E} \qquad \text{... (3.19)}$$

This is a strain equation in *x* direction.

Similarly, the strains in *y* and *z* directions can be calculated as follows:

$$\text{Strain in y direction, } e_y = -v\frac{\sigma_x}{E} + v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E} \qquad \text{... (3.20)}$$

$$\text{Strain in z direction, } e_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} + v\frac{\sigma_z}{E} \qquad \text{... (3.21)}$$

Solving equations (3.19), (3.20) and (3.21) for the normal stresses (σ_x, σ_y and σ_x), we get.

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[e_x(1-v) + ve_y + ve_z\right] \qquad \text{... (3.22)}$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)}\left[v\, e_x(1-v)e_y + ve_z\right] \qquad \text{... (3.23)}$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[v\, e_x + ve_y + (1-v)ve_z\right] \qquad \text{... (3.24)}$$

The shear stress and shear strain relationship is given by,

$$\tau = G\gamma$$

Where, $\tau \rightarrow$ Shear stress

$\gamma \rightarrow$ Shear strain

$G \rightarrow$ Modulus of rigidity or Shear modulus

The expressions for the three different sets of shear stresses are,

$$\tau_{xy} = G\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{zx} = G\gamma_{zx}$$

Where, $G \rightarrow$ Modulus of rigidity $= \dfrac{E}{2(1+v)}$

$$\Rightarrow \tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1-2v}{2}\right) \times \gamma_{xy} \qquad \text{... (3.25)}$$

$$\Rightarrow \tau_{yz} = \frac{E}{2(1+v)} \times \gamma_{yz}$$

$$\tau_{yz} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1-2v}{2}\right) \times \gamma_{yz} \qquad \text{... (3.26)}$$

$$\Rightarrow \tau_{zx} = \frac{E}{2(1+v)} \times \gamma_{yz}$$

$$\tau_{zx} = \frac{E}{(1+v)(1-2v)} \times \left(\frac{1-2v}{2}\right) \times \gamma_{zx} \qquad \text{... (3.27)}$$

Assembling the equations (3.22). (3.23), (3.24), (3.25), (3.26) and (3.27) in matrix form,

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{\mathrm{E}}{(1+\mathrm{v})(1-2\mathrm{v})} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad \ldots(3.28)$$

The above equation is in the form of

$$\{ \sigma \} = [D] \{ e \}$$

The above equation (3.28) gives a three dimensional stress-strain relationship for an isotropic body.

Where, [D] is stress-strain relationship matrix or constitutive matrix.

$$[D] = \frac{\mathrm{E}}{(1+\mathrm{v})(1-2\mathrm{v})} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \quad \ldots(3.29)$$

Where, E = Modulus of Elasticity or Young's modulus

V = Poisson's ratio

3.7.1 Plane Stress

For two dimensional plane stress problems, the normal stress, σ_z and shear stresses τ_{xy}, τ_{yz} are zero.

i.e., $\sigma_z = \tau_{xy} = \tau_{yz} = 0$

The shear strains γ_{xz}, γ_{yz} are zero but e_z are not equal to zero

i.e., $\gamma_{xz} = \gamma_{yz} = 0$

Substitute $\sigma_z = 0$ in equation (3.19),

$$\Rightarrow e_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} \quad \ldots(3.30)$$

Substitute $\sigma_z = 0$ in equation (3.20),

$$\Rightarrow e_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \text{... (3.31)}$$

Substitute in equation (3.30) and (3.31),

$$e_x = \frac{\sigma_x}{E} - \frac{v\sigma_y}{E}$$

$$v e_x = -v^2\frac{\sigma_x}{E} + v\frac{\sigma_y}{E} \qquad \text{[Equation (3.31)} \times v]$$

$$e_x + v e_y = \frac{\sigma_x}{E} - \frac{v^2\sigma_x}{E}$$

$$\Rightarrow e_x + v e_y = \frac{\sigma_x}{E}(1 - v^2)$$

$$\Rightarrow \qquad \sigma_x = \frac{E}{(1-v^2)}(e_x + v e_y) \qquad \text{... (3.32)}$$

Substitute in equation (3.30) and (3.31),

$$v\, e_x = v\frac{\sigma_x}{E} - v^2\frac{\sigma_y}{E} \qquad \text{[Equation (3.30)} \times v]$$

$$e_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$v\, e_x + e_y = v^2\frac{\sigma_y}{E} + v\frac{\sigma_y}{E}$$

$$v\, e_x + e_y = \frac{\sigma_y}{E}(1 - v^2)$$

$$\sigma_y = \frac{E}{(1-v^2)}(v\, e_x + v\, e_y) \qquad \text{... (3.33)}$$

We know that, Shear stress, $\tau_{xy} = G\,\gamma_{xy}$

Where, $\quad$ G → Modulus of rigidity = $\frac{E}{2(1+v)}$

γ_{xy} → Shear strain

v → Poisson's ratio

$$\Rightarrow \tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1-v)^2} \times \left(\frac{1-2v}{2}\right) \times \gamma_{zx} \qquad \text{... (3.34)}$$

Assembling the equations (3.32), (3.33) and (3.34) in matrix form,

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{\mathrm{E}}{(1-v)^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} \qquad \text{... (3.35)}$$

The above equation is in the form of

$$\{ \sigma \} = [D] \{ e \}$$

The above equation (3.35) gives a two dimensional stress-strain relationship for plane stress problems.

Where, [D] is stress-strain relationship matrix or constitutive matrix.

$$[D] = \frac{\mathrm{E}}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \qquad \text{... (3.36)}$$

Where, E = Modulus of Elasticity or Young's modulus

V = Poisson's ratio

3.7.2 Plane strain

For plane strain, we assume the following strains to be zero.

$$e_z = \gamma_{zx} = \gamma_{yz} = 0$$

The shear stresses $\tau_{xy} = \tau_{yz} = 0$, but $\sigma_z \neq 0$.

From equation (3.28), we know that,

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{\mathrm{E}}{(1+\mathrm{v})(1-2\mathrm{v})} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

In the above equation, e_z = 0, so, delete third row and third column of [D] matrix. γ_{yz}=0, so, delete fifth row and fifth column of [D] matrix. γ_{xy} =0, hence, delete sixth row and sixth column of [D] matrix. The final reduced equation is,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} \qquad \ldots (3.37)$$

The above equation is in the form of

$$\{ \sigma \} = [D] \{ e \}$$

The above equation (3.37) gives a two dimensional stress-strain relationship for plane strain problems.

Where, [D] = Stress-strain relationship or constitutive matrix.

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \qquad \ldots (3.36)$$

Where, E $\rightarrow$ Young's modulus

$v \rightarrow$ Poisson's ratio

3.8 STIFFNESS MATRIX EQUATION FOR TWO DIMENSIONAL ELEMENT (CST ELEMENT)

We know that,

$$\text{Stiffness matrix, [K]} = \int_v [B]^T [D][B] dV \qquad \text{[From chapter 2]}$$

$$[K] = [B]^T [D][B] V$$

$$\Rightarrow [K] = [B]^T [D][B] A\, t \qquad [\because V = A \times t]$$

$$\text{Stiffness matrix, [K]} = [B]^T [D][B] A\, t \qquad \ldots (3.39)$$

$$\text{Where, A} \longrightarrow \text{Area of the traingular element} = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix}$$

t → Thickness of element

[B] = Strain – Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \text{ [From equation no. (3.15)]}$$

Where $q_1 = y_2 - y_3;$ $q_2 = y_3 - y_1;$ $q_3 = y_1 - y_2$

$r_1 = x_3 - x_2;$ $r_2 = x_1 - x_3;$ $r_3 = x_2 - x_1$

[D] = Stress-Strain relationship matrix

For plane stress problems,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \text{ [From equation no. (3.36)]}$$

For plane strain problems,

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

[From equation no. (3.38)]

Where, E = Young's modulus or Modulus of Elasticity

V = Poisson's ratio

3.9. LINEAR STRAIN TRIANGULAR (LST) ELEMENT

A six noded triangular element is known as Linear Strain Triangular (LST) element which is shown in Fig.3.15. It has twelve unknown displacement degrees of freedom. The displacement functions of the element are quadratic instead of linear as in the CST.

The procedures for development of the stiffness matrix equations for the LST element follow the same steps as those used for the CST element. But the number of equations used for developing shift matrix equation is 12 instead of 6. It is a tedious process to solve those equations. Hence, we will use a computer, to solve many of the mathematical equations.

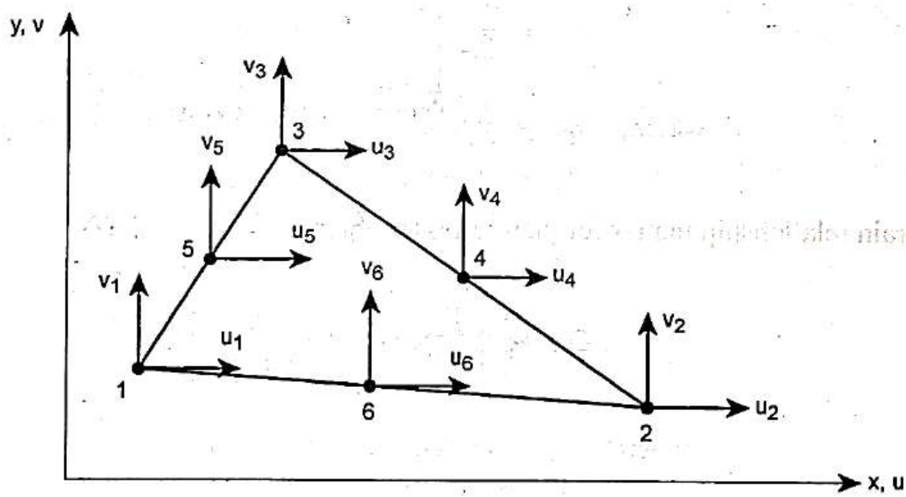


Fig. 3.15. Linear strain triangular element

LST element is preferred than the CST element for plane stress applications when relatively small numbers of nodes are used. LST element is not preferred when large numbers of nodes are used since the cost of formation of the element stiffnesses, equation bandwidth are high compared to CST element. Computer modelling for large number of nodes are also difficult for LST element.

3.10. FORMULAE USED

1. For constant strain triangle (CST) element

Shape function, $N_1 + N_2 + N_3 = 1$

Co-ordinate, $x = N_1 x + N_2 x_2 + N_3 x_3$

Co-ordinate, $y = N_1 y_1 + N_2 y_2 + N_3 y_3$

Or

Co-ordinate, $x = (x_1 - x_3)N_1 + (x_2 - x_3)N_2 + x_3$

Co-ordinate, $y = (y_1 - y_3)N_1 + (y_2 - y_3)N_2 + y_3$

2. Area of the traingular element, $A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix}$

3. Strain – Displacement matrix for CST element is,

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

Where $q_1 = y_2 - y_3;$ $\quad q_2 = y_3 - y_1;$ $\quad q_3 = y_1 - y_2$

$r_1 = x_3 - x_2;$ $\quad r_2 = x_1 - x_3;$ $\quad r_3 = x_2 - x_1$

4. Stress-Strain relationship matrix for plane stress problem,

$$[D] = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

Where, $v \rightarrow$ Poisson's ratio

E $\rightarrow$ Young's modulus

5. Stress-Strain relationship matrix for plane stress problem,

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

6. Element Stiffness matrix for CST element,

$$[K] = [B]^T[D][B]A\,t$$

7. Element stress, { σ } = [D] { e }

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\,D\,][\,B\,]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_2 \\ v_3 \end{Bmatrix}$$

Where, $\sigma_x, \sigma_y \rightarrow$ Normal stresses

$\tau_{xy} \rightarrow$ Shear stress

$u, v \rightarrow$ Nodal displacements

8. Maximum normal stress,

$$\sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum normal stress,

$$\sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

9. Principal angle,

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

10. Element strain,

$$\{e\} = [\,B\,]\{\,u\,\} = [\,B\,]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_2 \\ v_3 \end{Bmatrix}$$

3.11. SOLVED PROBLEMS-CST ELEMENTS

Example 3.1

Determine the shape functions N_1, N_2 and N_3 at the interior point P for the triangular element shown in Fig.(i).

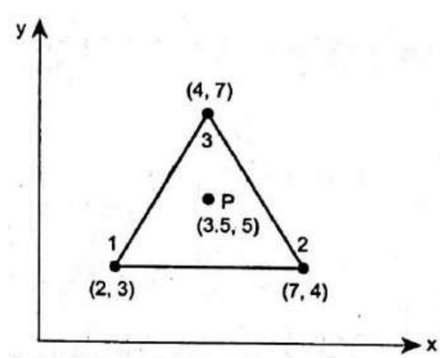


Fig. (i)

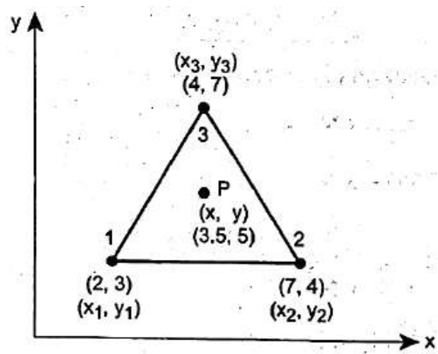
Given

$x_1 = 2, \ y_1 = 3$

$x_2 = 7, \ y_2 = 4$

$x_3 = 4, \ y_3 = 7$

$x = 3.5, \ y = 5$



To find: Shape functions N_1, N_2 and N_3 at the interior point, P.

Solution: We know that,

$$x = (x_1 - x_3)N_1 + (x_2 - x_3)N_2 + x_3 \quad \dots(1)$$

$$y = (y_1 - y_3)N_1 + (y_2 - y_3)N_2 + y_3 \quad \dots(2)$$

Substitute the co-ordinates values,

(1) ⇒ $$3.5 = (2 - 4)N_1 + (7 - 4)N_2 + 4 \quad \dots(3)$$

(2) ⇒ $$5 = (3 - 7)N_1 + (4 - 7)N_2 + 7 \quad \dots(4)$$

Equation (3) becomes,

$$3.5 = -2N_1 + 3N_2 + 4$$

⇒ $$-0.5 = -2N_1 + 3N_2$$

⇒ $$2N_1 - 3N_2 = 0.5 \quad \dots(5)$$

Equation (3) becomes, $5 = -4N_1 - 3N_2 + 7$

⇒ $$-2 = -4N_1 - 3N_2$$

⇒ $$4N_1 + 3N_2 = 2 \quad \dots(6)$$

Solving equation (5) and (6),

$$2N_1 - 3N_2 = 0.5$$

$$4N_1 + 3N_2 = 2$$

Solving $$6N_1 = 2.5$$

$$N_1 = 0.4166$$

Substituting N_1 value in equation (5) or equation (6),

$$2N_1 - 3N_2 = 0.5$$

$$2\times 0.4166 - 3N_2 = 2$$

$\Rightarrow$ $$N_2 = 0.11111$$

We know that, $N_1 + N_2\ N_3 = 1$

$\Rightarrow$ $$0.4166 + 0.1111 + N_3 = 1$$

$\Rightarrow$ $$N_3 = 0.4723$$

Result: Shape functions at the interior point, P.

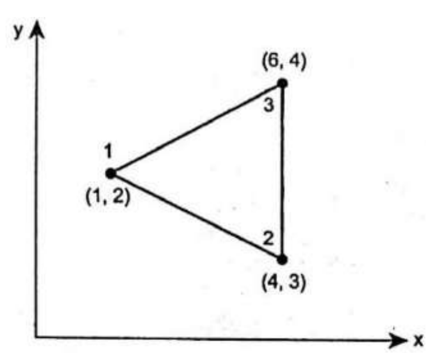
$$N_1 = 0.4166$$

$$N_2 = 0.11111$$

$$N_3 = 0.4723$$

Example 3.2

The nodal co-ordinates of the triangular element are shown in Fig.(i). At the interior point P, the x co-ordinate is 3.5 and $N_1 = 0.4$, calculate N_2, N_3 and the y co-ordinate at point P.



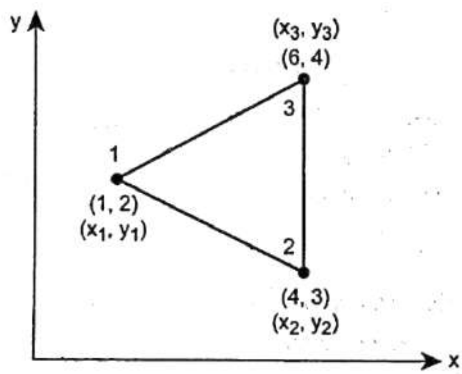
Given

$x_1 = 1,\ \ y_1 = 2$

$x_2 = 4,\ \ y_2 = 3$

$x_3 = 6,\ \ y_3 = 4$

$x = 3.5,\ \ y = 0.4$



To find: 1. Shape functions N_2 and N_3

2. Co-ordinate *y*.

Solution: We know that, Sum of shape function is equal to 1.

$$\Rightarrow N_1 + N_2 + N_3 = 1$$

$$\Rightarrow 0.4 + N_2 + N_3 = 1$$

$$\Rightarrow \quad N_2 + N_3 = 0.6$$

We know that, Co-ordinate, $x = N_1 x + N_2 x_2 + N_3 x_3$

$$3.5 = 0.4(1) + N_2 \times (4) + N_3 \times (6)$$

$$3.1 = 4\,N_2 + 6N_3$$

$$\Rightarrow 4N_2 + 6N_3 = 3.1 \qquad \ldots(2)$$

Solving equation (1) and (2,

$$-4N_2 - 4N_3 = -2.4$$

$$4N_2 + 6N_3 = 3.1$$

$$2N_3 = 0.7$$

$$N_3 = 0.35$$

Substituting N_3 value in equation (1) or (2),

$$N_2 + N_3 = 0.6$$

$$N_2 + 0.35 = 0.6$$

$$N_2 = 0.25$$

We know that, Co-ordinate, $y = N_1y_1 + N_2y_2 + N_3y_3$

$$= 0.4 \times 2 + 0.25 \times 3 + 0.35 \times 4$$

$$y = 2.95$$

Result: 1. Shape functions

$$N_2 = 0.11111$$

$$N_3 = 0.4723$$

2. Co-ordinate, $y = 2.95$

Example 3.3

Determine the x and y co-ordinates of point P for the triangular element shown in Fig.(i). The shape functions N_1 and N_2 are 0.2 and 0.3 respectively.

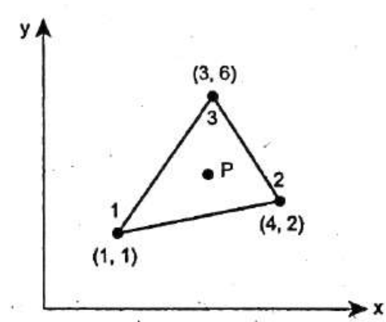
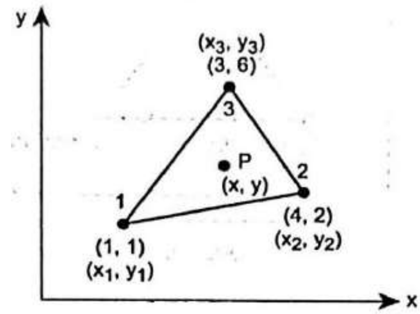


Fig. (i)

Given

$x_1 = 1,\quad y_1 = 1\quad x_2 = 4,\quad y_2 = 2$

$x_3 = 3, \quad y_3 = 6 \quad N_1 = 0.2, \quad N_2 = 0.3$



To find: *x* and *y* Co-ordinate of point P.

Solution: We know that,

Sum of shape function is equal to 1.

$$\Rightarrow N_1 + N_2 + N_3 = 1$$

$$\Rightarrow 0.2 + 0.3 + N_3 = 1$$

$$\Rightarrow \qquad N_3 = 0.5$$

x Co-ordinate at point P is,

$$x = N_1 x + N_2 x_2 + N_3 x_3$$

$$= 0.2(1) + 0.3 \times (4) + 0.5 \times (3)$$

$$x = 2.9$$

y Co-ordinate at point P is,

$$y = N_1 y + N_2 y_2 + N_3 y_3$$

$$= 0.2(1) + 0.3 \times (2) + 0.5 \times (6)$$

$$y = 3.8$$

Result: Co-ordinate of point P

x = 2.9

y = 3.8

Example 3.4

For the constant strain triangular element shown in Fig.(i), assemble strain-displacement matrix. Take t=20 mm and E = 2 x 10^5 N/mm².

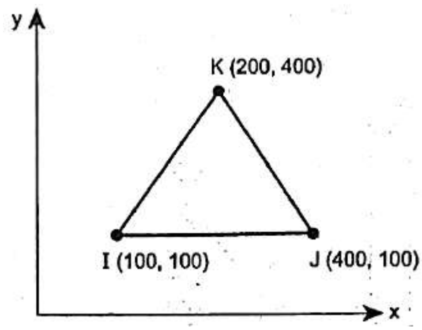


Fig. (i)

Given:

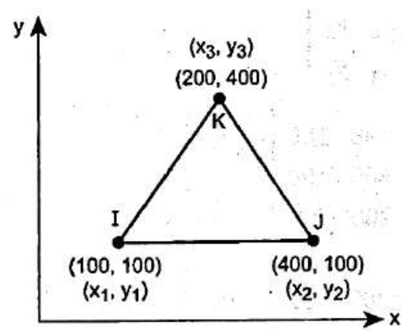


Fig. (ii)

$$x_1 = 100, \;\; y_1 = 100$$

$$x_2 = 400, \;\; y_2 = 100$$

$$x_3 = 200, \;\; y_3 = 400$$

Young's modulus, E = 2 × 10^5 N/mm^2

Thickness, t = 20 mm

To find: Strain – Displacement matrix [B]

Solution: We know that,

Strain – Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \quad \text{... (1)}$$

[From equation no. (3.15)]

Where $q_1 = y_2 - y_3 = 100 - 400 = -300$

$q_2 = y_3 - y_1 = 400 - 100 = 300$

$q_3 = y_1 - y_2 = 100 - 100 = 0$

$r_1 = x_3 - x_2 = 200 - 400 = -200$

$r_2 = x_1 - x_3 = 100 - 400 = -200$

$r_3 = x_2 - x_1 = 400 - 100 = 300$

Substitute the above values in equation (1)

$$\Rightarrow \quad [B] =$$

$$\frac{1}{2A}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix} \quad \text{... (2)}$$

Where, A = Area of the element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 100 & 100 \\ 1 & 400 & 100 \\ 0 & 200 & 400 \end{bmatrix}$$

$$= \frac{1}{2} \times [1(400 \times 400 - 200 \times 100) - 100(400 \times 1 - 100 \times 1) + 100(200 \times 1 - 400 \times 1)$$

A = 45, 000 mm^2

Substitute a value in equation (2),

$$\Rightarrow \quad [B] = \frac{1}{2A}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

Result: Strain – Displacement matrix

$$[B] = \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

Example 3.5

Determine the stiffness matrix for the constant strain triangular (CST) element shown in Fig.(i). The co-ordinates are given in units of millimeters. Assume plane stress conditions. Take E = 210 GPa, v=0.25 and t = 10 mm

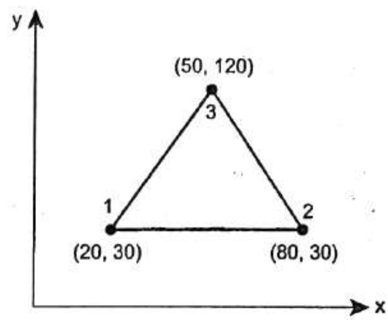


Fig. (i)

Given:

$$x_1 = 20, \;\; y_1 = 30$$

$$x_2 = 80, \;\; y_2 = 30$$

$$x_3 = 50, \;\; y_3 = 120$$

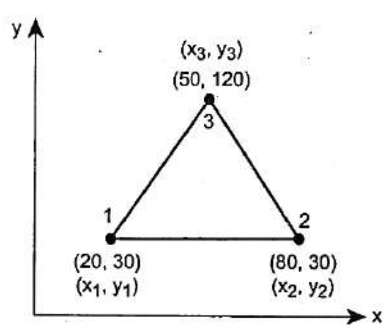


Fig. (ii)

Young's modulus, E = 210 Gpa = 2×10^9 Pa

$= 210 \times 10^9$ N/m^2 $= 210 \times 10^3$ N/mm^2

$E = 2.1 \times 10^5$ N/mm^2

Poisson's ratio $v = 0.25$

Thickness, t = 10 mm

Assume plane stress condition.

To find: Stiffness matrix [K]

Solution: We know that,

Stiffness matrix, [K] = [B]^T [D] [B] A t …(1)

[From equation no. (3.39)]

Where, A = Area of the element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 20 & 30 \\ 1 & 80 & 30 \\ 0 & 50 & 120 \end{bmatrix}$$

$$= \frac{1}{2} \times [1(80 \times 120 - 50 \times 30) - 20(120 - 30) + 30(50 - 80)]$$

$$= \frac{1}{2} \times [8100 - 1800 - 900]$$

A = 2700 mm^2 …(2)

We know that,

Strain – Displacement matrix

$$[B] = \frac{1}{2\text{A}}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \quad ...(3)$$

[From equation no. (3.15)]

Where $q_1 = y_2 - y_3 = 30 - 120 = -90$

$$q_2 = y_3 - y_1 = 120 - 30 = 90$$

$$q_3 = y_1 - y_2 = 30 - 30 = 0$$

$$r_1 = x_3 - x_2 = 50 - 80 = -30$$

$$r_2 = x_1 - x_3 = 20 - 50 = -30$$

$$r_3 = x_2 - x_1 = 80 - 20 = 60$$

Substitute the above values in equation (1)

$$\Rightarrow \qquad [B] = \frac{1}{2A}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

Substitute area, A value,

$$= \frac{1}{2 \times 2700}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$= \frac{30}{2 \times 2700}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B] = 5.555 \times 10^{-3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix} \qquad \dots (4)$$

We know that,

Stress-Strain relationship matrix [D] for plane stress problem is

$$[D] = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1 - v}{2} \end{bmatrix} \qquad \text{[From equation no. (3.36)]}$$

$$= \frac{2.1 \times 10^5}{1 - (0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1 - 0.25}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times 0.25}{0.9375} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\Rightarrow \quad [D][B] = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times 5.555$$

$$\times 10^{-3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$= 311.08 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$= 311.08 \begin{bmatrix} -12+0+0 & 0-1+0 & 12+0+0 & 0-1+0 & 0+0+0 & 0+2+0 \\ -3+0+0 & 0-4+0 & 3+0+0 & 0-4+0 & 0+0+0 & 0+8+0 \\ 0+0-1.5 & 0+0-4.5 & 0+0-1.5 & 0+0+4.5 & 0+0+3 & 0+0+0 \end{bmatrix}$$

$$Note: [(3 \times \underbrace{3) \times (3} \times 6) = 3 \times 6]$$

$$= 311.08 \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix} \qquad ...(6)$$

We know that,

$$[B] = 5.555 \times 10^{-3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix} \quad \text{[From equation no. 4]}$$

$$[B]^T = 5.555 \times 10^{-3} \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$\Rightarrow \quad [B]^T[D][B]$

$$= 5.555 \times 10^{-3} \begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\times 311.08 \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix}$$

$= 5.555 \times 10^{-3}$

$$\times 311.08 \begin{bmatrix} 36+0+1.5 & 3+1+4.5 & -36+0+1.5 & 3+0-4.5 & 0+0-3 & -6+0+0 \\ 0+3+4.5 & 0+4+13.5 & 0-3+4.5 & 0-4+0 & 0+0-9 & 0-8+0 \\ -36+0-1.5 & -3+0+4.5 & 36+0+1.5 & -3+0-4.5 & 0+0-3 & 6+0+0 \\ 0+3-4.5 & 0+4-13.5 & 0-3-4.5 & 0+4+13.5 & 0+0+9 & 0-8+0 \\ 0+0-3 & 0+0-9 & 0+0-3 & 0+0+9 & 0+0+6 & 0+0+0 \\ 0-6+0 & 0-8+0 & 0+6+0 & 0-8+0 & 0+0+0 & 0+16+0 \end{bmatrix}$$

$$[B]^T[D][B] = 1.78 \begin{bmatrix} 37.54 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ 34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

Substitute $[B]^T[D][B]$ and A, t values in equation (1),

Stiffness matrix [K]

$$= 1.78 \begin{bmatrix} 37.54 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ 34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix} \times 2700 \times 10 N/mm$$

$$[K] = 46.656 \times 10^3 \begin{bmatrix} 37.54 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ 34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix} N/mm$$

Result: Stiffness matrix [K]

$$[K] = 46.656 \times 10^3 \begin{bmatrix} 37.54 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ 34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix} N/mm$$

Example 3.6

For the plane stress element shown in Fig.(i), the nodal displacements are:

$u_1 = 2.0$ mm ; $v_1 = 1.0$ mm

$u_2 = 0.5$ mm ; $v_2 = 0.0$ mm

$u_3 = 3.0$ mm ; $v_3 = 1.0$ mm

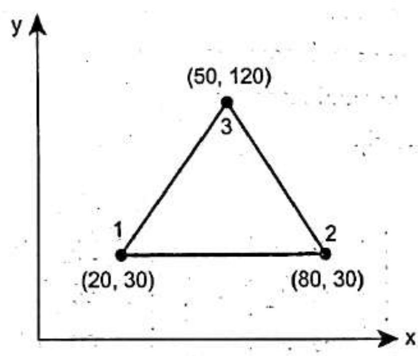


Fig. (i)

Determine the element stresses σ_x, σ_y, τ_{xy}, σ_1, and σ_2, and the principal angle θp Let E=210 GPa, v=0.25 and t = 10 mm. All co-ordinates are in millimeters.

Given: Nodal displacements:

$u_1 = 2.0$ mm ; $v_1 = 1.0$ mm

$u_2 = 0.5$ mm ; $v_2 = 0.0$ mm

$u_3 = 3.0$ mm ; $v_3 = 1.0$ mm

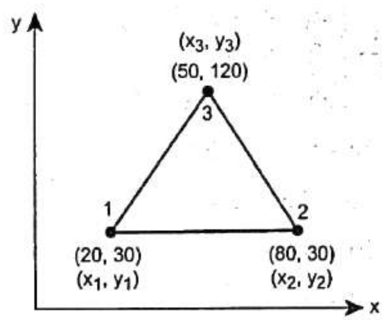


Fig. (ii)

$$x_1 = 20\ mm; \quad y_1 = 30\ mm$$

$$x_2 = 80\ mm \quad y_2 = 30\ mm$$

$$x_3 = 50mm \quad y_3 = 120\ \underline{mm}$$

Young's modulus, E = 210 GPa = 2×10^9 Pa

$= 210 \times 10^9$ N/m^2 $= 210 \times 10^3$ N/mm^2

E = 2.1×10^5 N/mm^2

Poisson's ratio $v = 0.25$

Thickness, t = 10 mm

To find: 1. Element stresses

(a) Normal stress, σ_x

(b) Normal stress, σ_y

(c) Shear stress, τ_{xy}

(d) Maximum normal stress, σ_1

(e) Minimum normal stress, σ_2

2. Principal angle, θ_P

Solution: We know that,

A = Area of the element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 20 & 30 \\ 1 & 80 & 30 \\ 0 & 50 & 120 \end{bmatrix}$$

$$= \frac{1}{2} \times [1(80 \times 120 - 50 \times 30) - 20(120 - 30) + 30(50 - 80)]$$

$$= \frac{1}{2} \times [8100 - 1800 - 900]$$

$$A = 2700 \text{ mm}^2 \qquad \ldots(1)$$

We know that,

Strain – Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad \ldots(2)$$

[From equation no. (3.15)]

Where $q_1 = y_2 - y_3 = 30 - 120 = -90$

$q_2 = y_3 - y_1 = 120 - 30 = 90$

$q_3 = y_1 - y_2 = 30 - 30 = 0$

$r_1 = x_3 - x_2 = 50 - 80 = -30$

$r_2 = x_1 - x_3 = 20 - 50 = -30$

$r_3 = x_2 - x_1 = 80 - 20 = 60$

Substitute the above values in equation (2)

$$\Rightarrow \qquad [B] = \frac{1}{2A}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

Substitute area, A value,

$$\Rightarrow \qquad [B] = \frac{1}{2 \times 2700}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$= \frac{30}{2 \times 2700}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B] = 5.555 \times 10^{-3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix} \qquad \ldots(3)$$

We know that,

Stress-Strain relationship matrix [D] for plane stress problem is

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \qquad \text{[From equation no. (3.36)]}$$

$$= \frac{2.1 \times 10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.25}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times 0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \qquad \ldots(4)$$

$$\Rightarrow \quad [D][B] = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times 5.555$$

$$\times 10^{-3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$= 311.08 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$= 311.08 \begin{bmatrix} -12+0+0 & 0-1+0 & 12+0+0 & 0-1+0 & 0+0+0 & 0+2+0 \\ -3+0+0 & 0-4+0 & 3+0+0 & 0-4+0 & 0+0+0 & 0+8+0 \\ 0+0-1.5 & 0+0-4.5 & 0+0-1.5 & 0+0+4.5 & 0+0+3 & 0+0+0 \end{bmatrix}$$

$$[D][B] = 311.08 \begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix} \qquad \ldots(5)$$

We know that,

stress, $\{ \sigma \} = [D] \{ e \}$

$$= [\,D\,][\,B\,]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$= 311.08\begin{bmatrix} -12 & -1 & 12 & -1 & 0 & 2 \\ -3 & -4 & 3 & -4 & 0 & 8 \\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0 \end{bmatrix} \times \begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 0.0 \\ 3 \\ 1.0 \end{Bmatrix}$$

$$= 311.08\begin{Bmatrix} (-12\times 2)+(-1\times 1)+(12\times 0.5)-(1\times 0)+(0\times 3)+(2\times 1) \\ (-3\times 2)-(4\times 1)+(3\times 0.5)-(4\times 0)+(0\times 8)+(8\times 1) \\ -(1.5\times 2)-(4.5\times 1)-(1.5\times 0.5)+(4.5\times 0)+(3\times 3)+(0\times 1) \end{Bmatrix}$$

$$Note: [(3\times \underbrace{6)\times(6}\times 1) = 3\times 1]$$

$$\{\,\sigma\,\} = 311.08\begin{Bmatrix} -17 \\ -0.5 \\ 0.75 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -5288.36 \\ -155.54 \\ 233.31 \end{Bmatrix} \qquad \ldots(6)$$

$\Rightarrow$ Normal stress, σ_x = -5288.36 N/mm^2

Normal stress, σ_y = -155.54 N/mm^2

Shear stress, τ_{xy} = 233.21 N/mm^2

We know that,

Maximum normal stress,

$$\sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \ldots(7)$$

$$= \frac{-5288.36 - 155.54}{2} + \sqrt{\left(\frac{-5288.36 - 155.54}{2}\right)^2 + (233.31)^2}$$

$$\sigma_1 = -144.956\ N/mm^2$$

Minimum normal stress,

$$\sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{... (7)}$$

$$= \frac{-5288.36 - 155.54}{2} + \sqrt{\left(\frac{-5288.36 - 155.54}{2}\right)^2 + (233.31)^2}$$

$$\sigma_2 = -5298.9\ N/mm^2$$

We know that, Principal angle,

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{... (9)}$$

$$\Rightarrow \quad 2\theta_P = \tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$

$$= \tan^{-1}\left[\frac{2 \times 233.31}{-5288.36 - 155.54}\right]$$

$$2\theta_P = -5.194^o$$

$$\Rightarrow \quad \theta_P = -2.59^o$$

Result: 1. Element stresses

(a) Normal stress, σ_x = - 5288.36 N/mm^2

(b) Normal stress, σ_y = - 155.54 N/mm^2

(c) Shear stress, τ_{xy} = 233.31 N/mm^2

(d) Maximum normal stress, σ_1 = - 144.956 N/mm^2

(e) Minimum normal stress, σ_2 = - 5298.9 N/mm^2

2. Principal angle, θ_P = -2.59^o

Example 3.7

For the triangular element shown in Fig.(i). Obtain the strain- displacement relation matrix [B] and determine the strains e_x, e_y and γ_{xy}

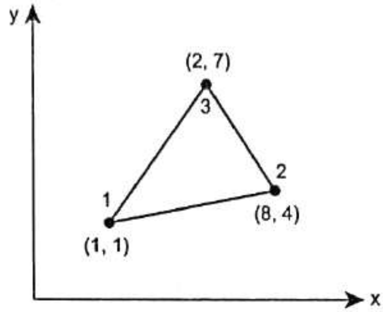


Fig.(i)

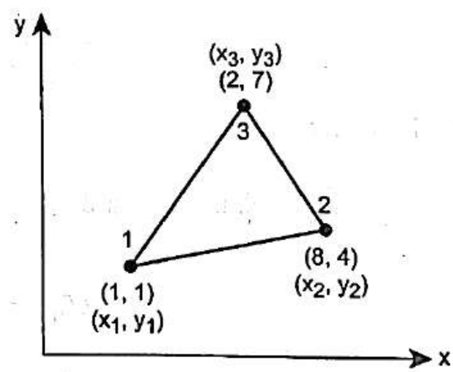
Nodal Displacements are

$u_1 = 0.001$; $v_1 = -0.004$

$u_2 = 0.003$; $v_2 = 0.002$

$u_3 = -0.002$; $v_3 = 0.005$ All coordinates are in millimetres

Given: Nodal Displacements:



$u_1 = 0.001$; $v_1 = -0.004$

$u_2 = 0.003$; $v_2 = 0.002$

$u_3 = -0.002$; $v_3 = 0.005$

$x_1 = 1\ mm;$ $y_1 = 1\ mm$

$x_2 = 8\ mm$ $y_2 = 4\ mm$

$x_3 = 2\ mm$ $y_3 = 7\ mm$

To find: Element stresses

(a) Normal stress, e_x

(b) Normal stress, e_y

(c) Shear stress, γ_{xy}

Solution: We know that,

A = Area of the element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 8 & 4 \\ 1 & 2 & 7 \end{bmatrix}$$

$$= \frac{1}{2} \times [1(56 - 8) - 1(7 - 4) + 1(2 - 8)]$$

A = 19.5 mm …(1)

We know that,

Strain – Displacement matrix

$$[B] = \frac{1}{2\text{A}} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \quad \text{... (2)}$$

Where $q_1 = y_2 - y_3 = 4 - 7 = -3$

$q_2 = y_3 - y_1 = 7 - 1 = 6$

$q_3 = y_1 - y_2 = 1 - 4 = -3$

$r_1 = x_3 - x_2 = 2 - 8 = -6$

$r_2 = x_1 - x_3 = 1 - 2 = -1$

$r_3 = x_2 - x_1 = 8 - 1 = 7$

Substitute the above values in equation (2)

$$\Rightarrow \quad [B] = \frac{1}{2\text{A}} \begin{bmatrix} -3 & 0 & 6 & 0 & -3 & 0 \\ 0 & -6 & 0 & -1 & 0 & 7 \\ -6 & -3 & -1 & 6 & 7 & -3 \end{bmatrix}$$

Substitute area, A value,

$$\Rightarrow \quad [B] = \frac{1}{2 \times 19.5} \begin{bmatrix} -3 & 0 & 6 & 0 & -3 & 0 \\ 0 & -6 & 0 & -1 & 0 & 7 \\ -6 & -3 & -1 & 6 & 7 & -3 \end{bmatrix}$$

We know that,

$$\text{Element strain,} \{ e \} = [B\,][\,u\,] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$= \frac{1}{39}\begin{bmatrix} -3 & 0 & 6 & 0 & -3 & 0 \\ 0 & -6 & 0 & -1 & 0 & 7 \\ -6 & -3 & -1 & 6 & 7 & -3 \end{bmatrix} \times \begin{Bmatrix} 0.001 \\ -0.004 \\ 0.003 \\ 0.002 \\ -0.002 \\ 0.005 \end{Bmatrix}$$

$$= \frac{1}{39}\begin{Bmatrix} (-3 \times 0.001) + 0 + (6 \times 0.003) + 0 + (3 \times 0.002) + 0 \\ 0 + (6 \times 0.004) + 0 - 1(1 \times 0.002) + 0 + (7 \times 0.005) \\ (-6 \times 0.001) + (3 + 0.004) - (1 \times 0.003) + (6 \times 0.002) + \big(7(-0.002)\big) + (-3 \times 0.005) \end{Bmatrix}$$

$$\{e\} = \frac{1}{39}\begin{Bmatrix} 0.0021 \\ 0.057 \\ -0.014 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 5.38 \times 10^{-4} \\ 1.4615 \times 10^{-3} \\ 3.589 \times 10^{-4} \end{Bmatrix}$$

$\Rightarrow$ $e_x = 5.38 \times 10^{-4}$

$e_y = 1.4615 \times 10^{-3}$

$\gamma_{xy} = 3.589 \times 10^{-4}$

Result: Element stresses

(a) Normal stress, $e_x = 5.38 \times 10^{-4}$

(b) Normal stress, $e_y = 1.4615 \times 10^{-3}$

(c) Shear stress, $\gamma_{xy} = 3.589 \times 10^{-4}$

Example 3.8

The two dimensional propped beam shown in Fig.(i) is divided into two CST elements. Determine the nodal displacements and element stresses using plane stress conditions. Body force is neglected in comparison with the external forces.

Take, Thickness, t = 10 mm

Young's modulus, E = 2 × 10^5 N/mm²

Poisson's ratio, v = 0.25

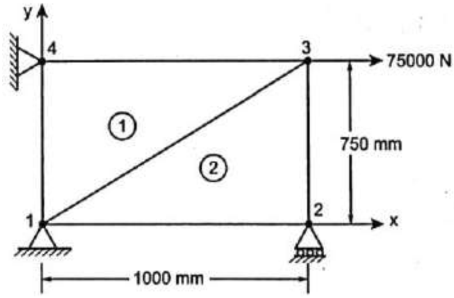


Fig. (i)

Given:

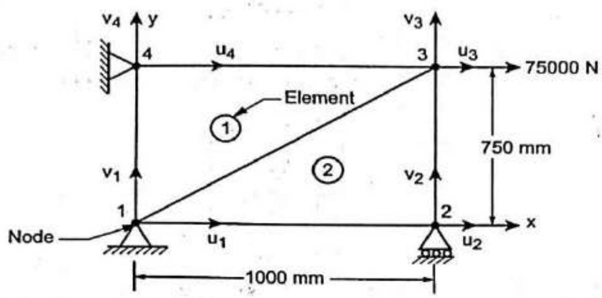


Fig. (ii)

Thickness, t = 10 mm

Young's modulus, E = 2×10^5 N/mm²

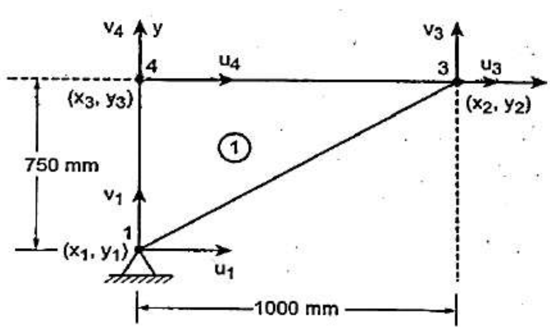
Poisson's ratio, v = 0.25

To find: (i) Nodal displacements $u_1 v_1, u_2 v_2, u_3, v_3$ and u_4, v_4

(ii) Element stress, σ_1 and σ_2

Solution:

Consider element (1): (Nodal displacements $u_1 v_1, u_3, v_3, u_4, v_4$)



Take node 1 as origin

For node 1: $\begin{matrix} x_1 & y_1 \\ (0, & 0) \end{matrix}$

For node 3: $\begin{matrix} x_2 & y_2 \\ (1000, & 750) \end{matrix}$

For node 4: $\begin{matrix} x_3 & y_3 \\ (0, & 750) \end{matrix}$

Solution: We know that,

Stiffness matrix [K] = $[B]^T$ [D] [B] A t …(1)

[From equation no. (3.39)]

Where, A = Area of the element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1000 & 750 \\ 1 & 0 & 750 \end{bmatrix}$$

$$= \frac{1}{2} \times 1 \times (1000 \times 750 - 0) = \frac{1000 \times 750}{2}$$

$$A = 375 \times 10^3 \text{ mm}^2 \quad \ldots(2)$$

Strain – Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \quad \ldots(3)$$

[From equation no. (3.15)]

Where $q_1 = y_2 - y_3 = 750 - 750 = 0$

$q_2 = y_3 - y_1 = 750 - 0 = 750$

$q_3 = y_1 - y_2 = 0 - 750 = -750$

$r_1 = x_3 - x_2 = 0 - 1000 = -1000$

$r_2 = x_1 - x_3 = 0 - 0 = 0$

$r_3 = x_2 - x_1 = 1000 - 0 = 1000$

Substitute the above values in equation (3)

$$\Rightarrow \qquad [B] = \frac{1}{2A}\begin{bmatrix} 0 & 0 & 750 & 0 & -750 & 0 \\ 0 & -1000 & 0 & 0 & 0 & 1000 \\ -1000 & 0 & 0 & 750 & 1000 & -750 \end{bmatrix}$$

Substitute area, A value,

$$\Rightarrow \qquad [B] = \frac{1}{2 \times 37510^3}\begin{bmatrix} 0 & 0 & 750 & 0 & -750 & 0 \\ 0 & -1000 & 0 & 0 & 0 & 1000 \\ -1000 & 0 & 0 & 750 & 1000 & -750 \end{bmatrix}$$

$$[B] = \frac{250}{2 \times 37510^3}\begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}$$

Stress-Strain relationship matrix [D] for plane stress problem is

$$[D] = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1 - v}{2} \end{bmatrix} \qquad \text{[From equation no. (3.36)]}$$

$$= \frac{2.1 \times 10^5}{1 - (0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1 - 0.25}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times 0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 2 \times 10^5 \times 0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \qquad \ldots(5)$$

$$\Rightarrow \qquad [D][B] = 2 \times 10^5 \times 0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\times \frac{250}{2 \times 375 \times 10^3}\begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}$$

$$= \frac{2 \times 10^5 \times 0.2667 \times 250}{2 \times 375 \times 10^3} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$= 17.78 \begin{bmatrix} 0+0+0 & 0-4+0 & 12+0+0 & 0+0+0 & -12+0+0 & 0+4+0 \\ 0+0+0 & 0-16+0 & 3+0+0 & 0+0+0 & -3+0+0 & 0+16+0 \\ 0+0 \mp & 0+0+0 & 0+0+0 & 0+0+4.5 & 0+0+6 & 0+0-4.5 \end{bmatrix}$$

$$[D][B] = 17.78 \begin{bmatrix} 0 & -4 & 12 & 0 & -12 & 4 \\ 0 & -16 & 3 & 0 & -3 & 16 \\ -6 & 0 & 0 & 4.5 & 6 & -4.5 \end{bmatrix} \qquad ...(6)$$

We know that,

$$[B] = \frac{250}{2 \times 37510^3} \begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix} \text{[From equation no. (4)]}$$

$$[B]^T = \frac{250}{2 \times 37510^3} \begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \\ -3 & 0 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

$$\Rightarrow [B]^T [D][B] = \frac{250}{2 \times 375 \times 10^3} \times 17.78 \begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \\ -3 & 0 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & 0 & 0 & 3 & 4 & -3 \end{bmatrix}$$

$$= 5.927 \times 10^3 \begin{bmatrix} 24 & 0 & 0 & -18 & -24 & 18 \\ 0 & 64 & -12 & 0 & 12 & -64 \\ 0 & -12 & 36 & 0 & -36 & 12 \\ -18 & 0 & 0 & 13.5 & 18 & -13.5 \\ -24 & 12 & -36 & 18 & 36+24 & -12-18 \\ 18 & -64 & 12 & -13.5 & -12-18 & +64+13.5 \end{bmatrix}$$

$$[B]^T [D][B] = 5.927 \times 10^3 \begin{bmatrix} 24 & 0 & 0 & -18 & -24 & 18 \\ 0 & 64 & -12 & 0 & 12 & -64 \\ 0 & -12 & 36 & 0 & -36 & 12 \\ -18 & 0 & 0 & 13.5 & 18 & -13.5 \\ -24 & 12 & -36 & 18 & 60 & -30 \\ 18 & -64 & 12 & -13.5 & -30 & 77.5 \end{bmatrix}$$

Substitute $[B]^T\ [D][B]$ and A, t values in equation (1),

Stiffness matrix, $[K]$

$$= 5.927 \times 10^3 \begin{bmatrix} 24 & 0 & 0 & -18 & -24 & 18 \\ 0 & 64 & -12 & 0 & 12 & -64 \\ 0 & -12 & 36 & 0 & -36 & 12 \\ -18 & 0 & 0 & 13.5 & 18 & -13.5 \\ -24 & 12 & -36 & 18 & 60 & -30 \\ 18 & -64 & 12 & -13.5 & -30 & 77.5 \end{bmatrix} \times 375 \times 10^3 \times 10$$

$$= 2.22 \times 10^4 \begin{bmatrix} 24 & 0 & 0 & -18 & -24 & 18 \\ 0 & 64 & -12 & 0 & 12 & -64 \\ 0 & -12 & 36 & 0 & -36 & 12 \\ -18 & 0 & 0 & 13.5 & 18 & -13.5 \\ -24 & 12 & -36 & 18 & 60 & -30 \\ 18 & -64 & 12 & -13.5 & -30 & 77.5 \end{bmatrix}$$

$$[K] = 1 \times 10^4 \begin{bmatrix} 53.28 & 0 & 0 & -39.96 & -53.28 & 39.96 \\ 0 & 142.08 & -26.64 & 0 & 26.64 & -142.08 \\ 0 & -26.64 & 79.92 & 0 & -79.92 & 26.64 \\ -39.96 & 0 & 0 & 29.97 & 39.96 & -29.97 \\ -53.28 & 26.64 & -79.92 & 39.96 & 133.2 & -66.6 \\ 39.96 & -142.08 & 26.64 & -29.97 & -66.6 & 172.05 \end{bmatrix}$$

For element (1), nodal displacements are $u_1 v_1, u_3, v_3, u_4, v_4$. [Refer Fig. (iii)]

$$\text{Stiffness matrix, } [K] = 1 \times 10^4 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 24 & 0 & 0 & -18 & -24 & 18 \\ 0 & 64 & -12 & 0 & 12 & -64 \\ 0 & -12 & 36 & 0 & -36 & 12 \\ -18 & 0 & 0 & 13.5 & 18 & -13.5 \\ -24 & 12 & -36 & 18 & 60 & -30 \\ 18 & -64 & 12 & -13.5 & -30 & 77.5 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_2 \\ v_3 \end{array}$$

…(7)

Consider element (2): (Nodal displacements $u_1 v_1, u_2, v_2,$ and u_3, v_3)

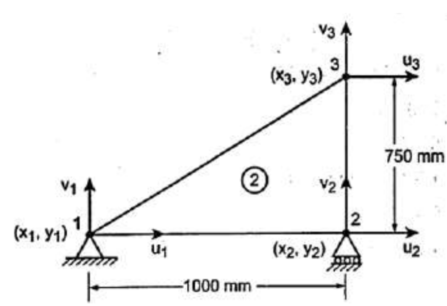


Fig. (iv)

Take node 1 as origin

$\qquad\qquad\quad x_1 \;\; y_1$

For node 1: (0, 0)

$\qquad\qquad\quad x_2 \qquad y_2$

For node 2: (1000, 0)

$\qquad\qquad\quad x_3 \qquad y_3$

For node 3: (1000, 750)

We know that,

Stiffness matrix for element (2) $[\,K\,]_2 = [B]^T\,[D]\,[B]\,A\,t$ …(8)

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1000 & 0 \\ 1 & 0 & 750 \end{bmatrix}$$

$$= \frac{1}{2} \times 1 \times (1000 \times 750 - 0) = \frac{1000 \times 750}{2}$$

$A = 375 \times 10^3\ mm^2$

Strain – Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad \text{... (9)}$$

Where $q_1 = y_2 - y_3 = 0 - 750 = -750$

$q_2 = y_3 - y_1 = 750 - 0 = 750$

$q_3 = y_1 - y_2 = 0 - 0 = 0$

$r_1 = x_3 - x_2 = 1000 - 1000 = 0$

$r_2 = x_1 - x_3 = 0 - 1000 = -1000$

$r_3 = x_2 - x_1 = 1000 - 0 = 1000$

$$\Rightarrow \quad [B] = \frac{1}{2A}\begin{bmatrix} -750 & 0 & 750 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1000 & 0 & 1000 \\ 0 & -750 & -1000 & 750 & 1000 & 0 \end{bmatrix}$$

Substitute area, A value,

$$\Rightarrow \qquad [B] = \frac{1}{2 \times 375 \times 10^3} \begin{bmatrix} -750 & 0 & 750 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1000 & 0 & 1000 \\ 0 & -750 & -1000 & 750 & 1000 & 0 \end{bmatrix}$$

$$[B] = \frac{250}{2 \times 375 \times 10^3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

Stress-Strain relationship matrix [D] for plane stress problem is

$$[D] = \frac{\mathrm{E}}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1 - v}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{1 - (0.25)^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1 - 0.25}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times}{0.9375} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{2.1 \times 10^5 \times 0.25}{0.9375} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \qquad \ldots (11)$$

$$[D][B] = \frac{250 \times 2 \times 10^5 \times 0.25}{2 \times 375 \times 10^3 \times 0.9375} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

$$= 17.78 \begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix} \qquad \ldots (12)$$

We know that,

$$[B] = \frac{250}{2 \times 375 \times 10^3} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix} \text{[From equation no. (10)]}$$

$$[B]^T = \frac{250}{2 \times 375 \times 10^3}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 3 & 0 & -4 \\ 0 & -4 & 3 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = \frac{250}{2 \times 37510^3}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 3 & 0 & -4 \\ 0 & -4 & 3 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix}$$

$$= 5.927 \times 10^3 \begin{bmatrix} 36 & 0 & -36 & 12 & 0 & -12 \\ 0 & 13.5 & 18 & -13.5 & -18 & 0 \\ -36 & 18 & 60 & -30 & -24 & 12 \\ 12 & -13.5 & -30 & 77.5 & 18 & -64 \\ 0 & -18 & -24 & 18 & 24 & 0 \\ 12 & 0 & 12 & -64 & 0 & 64 \end{bmatrix}$$

Substitute $[B]^T$ $[D][B]$ and A, t values in equation (8),

$$\text{Stiffness matrix, } [K]_2 = 5.927 \times 10^3 \begin{bmatrix} 36 & 0 & -36 & 12 & 0 & -12 \\ 0 & 13.5 & 18 & -13.5 & -18 & 0 \\ -36 & 18 & 60 & -30 & -24 & 12 \\ 12 & -13.5 & -30 & 77.5 & 18 & -64 \\ 0 & -18 & -24 & 18 & 24 & 0 \\ 12 & 0 & 12 & -64 & 0 & 64 \end{bmatrix} \times 375 \times 10^3 \times 10$$

$$= 2.22 \times 10^4 \begin{bmatrix} 36 & 0 & -36 & 12 & 0 & -12 \\ 0 & 13.5 & 18 & -13.5 & -18 & 0 \\ -36 & 18 & 60 & -30 & -24 & 12 \\ 12 & -13.5 & -30 & 77.5 & 18 & -64 \\ 0 & -18 & -24 & 18 & 24 & 0 \\ 12 & 0 & 12 & -64 & 0 & 64 \end{bmatrix}$$

$$[K] = 1 \times 10^4 \begin{bmatrix} 79.92 & 0 & -79.92 & 26.64 & 0 & -26.64 \\ 0 & 29.97 & 39.96 & -29.97 & -39.96 & 0 \\ -79.92 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 \\ 26.64 & -29.97 & -66.6 & 172.05 & 39.96 & -142.08 \\ 0 & -39.96 & -53.28 & 39.96 & 53.28 & 0 \\ -26.64 & 0 & 26.64 & -142.08 & 0 & 142.08 \end{bmatrix}$$

For element (2), nodal displacements are $u_1 v_1,\ u_2, v_2$ and u_3, v_3 [Refer Fig. (iv)]

Stiffness matrix, $[K]_2$

$$
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\
= 1 \times 10^4 \begin{bmatrix} 79.92 & 0 & -79.92 & 26.64 & 0 & -26.64 \\ 0 & 29.97 & 39.96 & -29.97 & -39.96 & 0 \\ -79.92 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 \\ 26.64 & -29.97 & -66.6 & 172.05 & 39.96 & -142.08 \\ 0 & -39.96 & -53.28 & 39.96 & 53.28 & 0 \\ -26.64 & 0 & 26.64 & -142.08 & 0 & 142.08 \end{bmatrix}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
\end{array}
\quad \ldots(13)
$$

Global stiffness Matrix [K]

Assemble the stiffness matrix equations (7) and (13),

Global stiffness Matrix [K] =

1×10^4

u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	
53.28 +79.92	0 + 0	-79.92	26.64	0 + 0	-39.96 -26.64	-53.28	39.96	u_1
0 + 0	+ 142.08 + 29.97	39.96	-29.97	-26.64 + -39.96	0 + 0	26.64	-142.08	v_1
-79.92	39.96	133.2	-66.6	-53.28	26.64	0	0	u_2
26.64	-29.97	-66.6	172.05	39.96	-142.08	0	0	v_2
0 + 0	-26.64 + -39.96	-53.28	39.96	79.92 + 53.28	0 + 0	-79.92	26.64	u_3
-39.96 + -26.64	0 + 0	26.64	-142.08	0 + 0	29.97 + 142.08	39.96	-29.97	v_3
-53.28	26.64	0	0	-79.92	39.96	133.2	-66.6	u_4
39.96	-142.08	0	0	26.64	-29.97	-66.6	172.05	v_4

Global Stiffness matrix, $[K] =$

$$1\times 10^4 \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 133.2 & 0 & -79.92 & 26.64 & 0 & -66.6 & -53.28 & 39.96 \\ 0 & 172.05 & 39.96 & -29.97 & -66.6 & 0 & 26.64 & -142.08 \\ -79.92 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 & 0 & 0 \\ 26.64 & -29.97 & -66.6 & -53.28 & 26.64 & -142.08 & 0 & 0 \\ 0 & -66.6 & -53.28 & 39.96 & 133.2 & 0 & -79.92 & 26.64 \\ -66.6 & 0 & 26.64 & -142.08 & 0 & 172.08 & 39.96 & -29.97 \\ -53.28 & 26.64 & 0 & 0 & -79.92 & 39.96 & 133.2 & -66.6 \\ 39.96 & -142.08 & 0 & 0 & 26.64 & -29.97 & -66.6 & 172.05 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_2 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

We Know that, general force equation is

{ F } = [K] {u }

$$\begin{Bmatrix} F_{1-x} \\ F_{1-y} \\ F_{2-x} \\ F_{2-y} \\ F_{3-x} \\ F_{3-y} \\ F_{4-x} \\ F_{4-y} \end{Bmatrix} = 1\times 10^4 \begin{bmatrix} 133.2 & 0 & -79.92 & 26.64 & 0 & -66.6 & -53.28 & 39.96 \\ 0 & 172.05 & 39.96 & -29.97 & -66.6 & 0 & 26.64 & -142.08 \\ -79.92 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 & 0 & 0 \\ 26.64 & -29.97 & -66.6 & -53.28 & 26.64 & -142.08 & 0 & 0 \\ 0 & -66.6 & -53.28 & 39.96 & 133.2 & 0 & -79.92 & 26.64 \\ -66.6 & 0 & 26.64 & -142.08 & 0 & 172.08 & 39.96 & -29.97 \\ -53.28 & 26.64 & 0 & 0 & -79.92 & 39.96 & 133.2 & -66.6 \\ 39.96 & -142.08 & 0 & 0 & 26.64 & -29.97 & -66.6 & 172.05 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \ldots(14)$$

Applying boundary conditions [Refer Fig. (ii)]

1. Node, 1 and Node 4 are fixed. So, u_1, v_1 and u_4, v_4 are zero. i.e., $u_1 = v_1 = u_4 = v_4 = 0$.
2. Node 2, is moving in *x* direction. So $u_2 \neq 0$ but, $v_2 = 0$.
3. At node 3, a point load of 75000N is acting in *x* direction. So, $F_{3\text{-}x}$ = 75,000N.
4. Body force is neglected. So, the remaining forces are zero.

   i.e., $F_{1x} = F_{1y} = F_{2x} = F_{2y} = F_{3x} = F_{3y} = F_{4x} = F_{4y} = 0.$

Substitute the above values in equation (14),

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 75\times 10^3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 1\times 10^4 \begin{bmatrix} 133.2 & 0 & -79.92 & 26.64 & 0 & -66.6 & -53.28 & 39.96 \\ 0 & 172.05 & 39.96 & -29.97 & -66.6 & 0 & 26.64 & -142.08 \\ -79.92 & 39.96 & 133.2 & -66.6 & -53.28 & 26.64 & 0 & 0 \\ 26.64 & -29.97 & -66.6 & -53.28 & 26.64 & -142.08 & 0 & 0 \\ 0 & -66.6 & -53.28 & 39.96 & 133.2 & 0 & -79.92 & 26.64 \\ -66.6 & 0 & 26.64 & -142.08 & 0 & 172.08 & 39.96 & -29.97 \\ -53.28 & 26.64 & 0 & 0 & -79.92 & 39.96 & 133.2 & -66.6 \\ 39.96 & -142.08 & 0 & 0 & 26.64 & -29.97 & -66.6 & 172.05 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ v_3 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation u_1, v_1, v_2, u_4, v_4 are zero. So delete the corresponding row and column of [K] matrix. Hence the equation reduces to

$$\begin{Bmatrix} 0 \\ 75 \times 10^3 \\ 0 \end{Bmatrix} = 1 \times 10^4 \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ -53.28 & 133.2 & 0 \\ 26.64 & 0 & 172.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ 7.5 \\ 0 \end{Bmatrix} = \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ -53.28 & 133.2 & 0 \\ 26.64 & 0 & 172.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad \ldots(15)$$

$$\Rightarrow \begin{Bmatrix} 0 \\ 18.75 \\ 18.75 \end{Bmatrix} = \begin{bmatrix} 133.2 & -53.28 & 26.64 \\ 0 & 279.72 & 0 \\ 0 & 0 & 172.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad -4349.81\, v_3 = 18.75$$

$$\Rightarrow \quad v_3 = -0.00431\, mm$$

$$\Rightarrow \quad 279.72 u_3 + 26.64 v_3 = 18.75$$

$$\Rightarrow \quad 279.72 u_3 + 26.64 \times (-0.00431) = 18.75$$

$$\Rightarrow \quad u_3 = 0.067\, mm$$

$$\Rightarrow \quad 133.2 u_3 - 53.28 u_3 + 26.64 v_3 = 0$$

$$\Rightarrow \quad 133.2 u_2 - 53.28(0.067) + 26.64(-0.00431) = 0$$

$$\Rightarrow \quad u_2 = 0.02766\, mm$$

Nodal displacements:

$u_1 = 0 mm$ $\qquad v_1 = 0\, mm$

$u_2 = 0.02766 mm$ $\qquad v_2 = 0\, mm$

$u_3 = 0.067\, mm$ $\qquad v_3 = -0.00431\, mm$

$u_4 = 0\, mm$ $\qquad v_4 = 0\, mm$

Stress in each element:

We know that, stress, { σ } = [D] [B] { u }

For element (1): [Refer Fig. (iii)].: (Nodal displacements equation u_1, v_1, u_3, v_3 and u_4, v_4)

$$\Rightarrow \text{ stress}, \{\sigma\}_1 = 17.78 \begin{bmatrix} 0 & -4 & 12 & 0 & -12 & 4 \\ 0 & -16 & 3 & 0 & -3 & 16 \\ -6 & 0 & 0 & 4.5 & 6 & -4.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

[From equation no. (6)]

$$= 17.78 \begin{bmatrix} 0 & -4 & 12 & 0 & -12 & 4 \\ 0 & -16 & 3 & 0 & -3 & 16 \\ -6 & 0 & 0 & 4.5 & 6 & -4.5 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0.067 \\ -0.00431 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\sigma\}_1 = 17.78 \begin{Bmatrix} 12 \times 0.067 \\ 3 \times 0.067 \\ 4.5 \times -0.00431 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 14.295 \\ 3.574 \\ -0.345 \end{Bmatrix} N/mm^2$$

$\Rightarrow$ Where, $\sigma_x, \sigma_y \rightarrow$ Normal stress,

$\tau_{xy} \rightarrow$ Shear stress

For element (2): [Refer Fig. (iv)].: (Nodal displacements equation u_1, v_1, u_2, v_2 and u_3, v_3)

Stress, $\{\sigma_2\} = [D]_2 [B]_2 \{u\}$

$$= 17.78 \begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

[From equation no. (12)]

$$= 17.78 \begin{bmatrix} -12 & 0 & 12 & -4 & 0 & 4 \\ -3 & 0 & 3 & -16 & 0 & 16 \\ 0 & -4.5 & -6 & 4.5 & 6 & 0 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0.02766 \\ 0 \\ 0.067 \\ -0.00431 \end{Bmatrix}$$

$$\{\sigma\}_2 = 17.78 \begin{Bmatrix} (12 \times 0.02766) + (4 \times -0.00431) \\ (3 \times 0.02766) + (16 \times -0.00431) \\ (-6 \times -0.00431) + (6 \times 0.067) \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 5.595 \\ 0.2492 \\ 4.196 \end{Bmatrix} N/mm^2$$

Result:

1. Nodal displacements: $u_1 = 0, v_1 = 0, u_2 = 0.2766mm; v_2 = 0, u_3 = 0.067mm;$ $v_3 = -0.00431\ mm; u_4 = 0\ and\ v_4 = 0.$
2. For element (1) $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 14.295 \\ 3.574 \\ -0.345 \end{Bmatrix} N/mm^2$
3. For element (2) $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 5.595 \\ 0.2492 \\ 4.196 \end{Bmatrix} N/mm^2$

Example 3.9

Determine the nodal displacements of nodes 1 and 2 and the element stresses for the two dimensional loaded plate as shown in Fig.(i). Assume plane stress condition. Take, v= 0.25, E = 2 x 10^5 N/mm², Thickness = 15 mm.

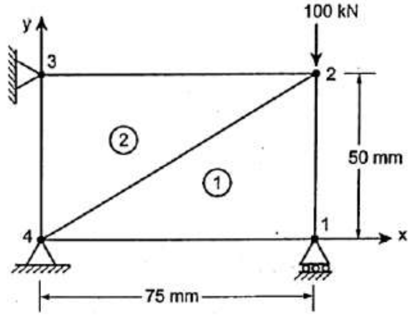


Fig. (i)

Body force is neglected in comparison with external forces.

Given:

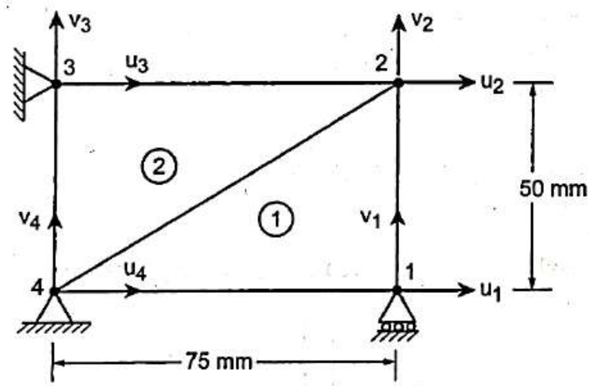


Fig. (ii)

Thickness, t = 10 mm

Young's modulus, E = 2 × 10^5 N/mm²

Poisson's ratio, v = 0.25

Point load acting at node 2 is 100 kN.

To find: (i) Nodal displacements at node 1 and 2. i.e., $u_1 v_1$, and $u_2 v_2$

(ii) Element stress, σ_1 and σ_2

Solution:

Consider element (1): (Nodal displacements $u_1 v_1, u_2, v_2, u_4, v_4$)

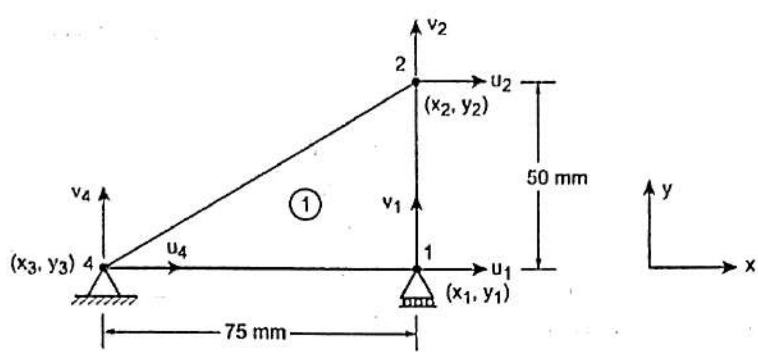


Fig. (iii)

Take node 1 as origin

$$\begin{array}{ll} & x_1 \;\; y_1 \\ \text{For node 1:} & (0, \;\; 0) \end{array}$$

$$\begin{array}{ll} & x_2 \;\; y_2 \\ \text{For node 3:} & (0, \;\; 50) \end{array}$$

$$\begin{array}{ll} & x_3 \;\; y_3 \\ \text{For node 3:} & (-75, \;\; 0) \end{array}$$

Solution: We know that,

Stiffness matrix [K] = $[B]^T$ [D] [B] A t …(1)

[From equation no. (3.39)]

Where, A = Area of the element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 50 \\ 1 & -75 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \times 1 \times (0 \times 75 \times 50) = \frac{3750}{2}$$

A = 1875 mm^2 …(2)

Strain – Displacement matrix

$$[B] = \frac{1}{2\text{A}}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad \text{... (3)}$$

[From equation no. (3.15)]

Where $q_1 = y_2 - y_3 = 50 - 0 = 50$

$q_2 = y_3 - y_1 = 0 - 0 = 0$

$q_3 = y_1 - y_2 = 0 - 50 = -50$

$r_1 = x_3 - x_2 = -75 - 0 = -75$

$r_2 = x_1 - x_3 = 0 + 75 = 75$

$r_3 = x_2 - x_1 = 0 - 0 = 0$

Substitute the above values in equation (3)

$$\Rightarrow \qquad [B]=\frac{1}{2A}\begin{bmatrix} 50 & 0 & 0 & 0 & -50 & 0 \\ 0 & -75 & 0 & 75 & 0 & 0 \\ -75 & 50 & 75 & 0 & 0 & -50 \end{bmatrix}$$

Substitute area, A value,

$$\Rightarrow \qquad [B]=\frac{1}{2\times 1875}\begin{bmatrix} 50 & 0 & 0 & 0 & -50 & 0 \\ 0 & -75 & 0 & 75 & 0 & 0 \\ -75 & 50 & 75 & 0 & 0 & -50 \end{bmatrix}$$

$$[B]=\frac{50}{3750}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1.5 & 0 & 1.5 & 0 & 0 \\ -1.5 & 1 & 1.5 & 0 & 0 & -1 \end{bmatrix} \qquad \text{... (4)}$$

Stress-Strain relationship matrix [D] for plane stress problem is

$$[D]=\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \qquad \text{[From equation no. (3.36)]}$$

$$=\frac{2\times 10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.25}{2} \end{bmatrix}$$

$$=\frac{2\times 10^5}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$=\frac{2\times 10^5\times 0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D]=2\times 10^5\times 0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \qquad \text{...(5)}$$

$$\Rightarrow \qquad [D][B]=2\times 10^5\times 0.2667\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\times\frac{50}{3750}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1.5 & 0 & 1.5 & 0 & 0 \\ -1.5 & 1 & 1.5 & 0 & 0 & -1 \end{bmatrix}$$

$$= \frac{2 \times 10^5 \times 0.2667 \times 50}{3750} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1.5 & 0 & 1.5 & 0 & 0 \\ -1.5 & 1 & 1.5 & 0 & 0 & -1 \end{bmatrix}$$

$$= 711.2 \begin{bmatrix} 4 & -1.5 & 0 & 1.5 & -4 & 0 \\ 1 & -4 \times 1.5 & 0 & 4 \times 1.5 & -1 & 0 \\ 1.5 \times -1.5 & 1.5 & 1.5 \times -1.5 & 0 & 0 & -1.5 \end{bmatrix}$$

$$[D][B] = 711.2 \begin{bmatrix} 4 & -1.5 & 0 & 1.5 & -4 & 0 \\ 1 & -6 & 0 & 6 & -1 & 0 \\ -2.25 & 1.5 & 2.25 & 0 & 0 & -1.5 \end{bmatrix} \qquad \dots (6)$$

We know that,

$$[B] = \frac{50}{3750} \begin{bmatrix} 1 & 0 & 3 & 0 & -1 & 0 \\ 0 & -1.5 & 0 & 1.5 & 0 & 0 \\ -1.5 & 1 & 1.5 & 0 & 0 & -1 \end{bmatrix} \text{[From equation no. (4)]}$$

$$[B]^T = \frac{50}{3750} \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & -1.5 & 1 \\ 0 & 0 & 1.5 \\ 0 & 1.5 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow [B]^T [D][B] = \frac{50}{3750} \times 711.2 \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & -1.5 & 1 \\ 0 & 0 & 1.5 \\ 0 & 1.5 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & -1.5 & 0 & 1.5 & -4 & 0 \\ 1 & -6 & 0 & 6 & -1 & 0 \\ -2.25 & 1.5 & 2.25 & 0 & 0 & -1.5 \end{bmatrix}$$

$$[B]^T [D][B] = 9.483 \begin{bmatrix} 7.375 & -3.75 & -3.375 & 1.5 & -4 & 2.25 \\ -3.75 & 10.5 & 2.25 & -9 & 1.5 & -1.5 \\ -3.375 & 2.25 & 3.375 & 0 & 0 & -2.25 \\ 1.5 & -9 & 0 & 9 & -1.5 & 0 \\ -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix}$$

Substitute $[B]^T [D][B]$ and A, t values in equation (1),

Stiffness matrix, $[K]$

$$= 9.483 \begin{bmatrix} 7.375 & -3.75 & -3.375 & 1.5 & -4 & 2.25 \\ -3.75 & 10.5 & 2.25 & -9 & 1.5 & -1.5 \\ -3.375 & 2.25 & 3.375 & 0 & 0 & -2.25 \\ 1.5 & -9 & 0 & 9 & -1.5 & 0 \\ -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \times 1875 \times 15$$

$$= 26.67 \times \begin{bmatrix} 7.375 & -3.75 & -3.375 & 1.5 & -4 & 2.25 \\ -3.75 & 10.5 & 2.25 & -9 & 1.5 & -1.5 \\ -3.375 & 2.25 & 3.375 & 0 & 0 & -2.25 \\ 1.5 & -9 & 0 & 9 & -1.5 & 0 \\ -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix}$$

$$[K] = 1 \times 10^4 \begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 40.01 & -40.01 \\ -90.01 & 60 & 90.01 & 0 & 0 & -60 \\ 40.01 & -240.03 & 0 & 240.03 & -40.01 & 0 \\ -106.68 & 40.01 & 0 & -40.01 & 106.68 & 0 \\ 60 & -40.01 & -60 & 0 & 0 & 40.01 \end{bmatrix}$$

For element (1), nodal displacements are $u_1 v_1, u_2, v_2, u_4, v_4$. [Refer Fig. (iii)]

Stiffness matrix, $[K] =$

$$= 1 \times 10^4 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 40.01 & -40.01 \\ -90.01 & 60 & 90.01 & 0 & 0 & -60 \\ 40.01 & -240.03 & 0 & 240.03 & -40.01 & 0 \\ -106.68 & 40.01 & 0 & -40.01 & 106.68 & 0 \\ 60 & -40.01 & -60 & 0 & 0 & 40.01 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{array}$$

...(7)

Consider element (2): (Nodal displacements $u_3 v_3, u_4, v_4,$ and u_2, v_2)

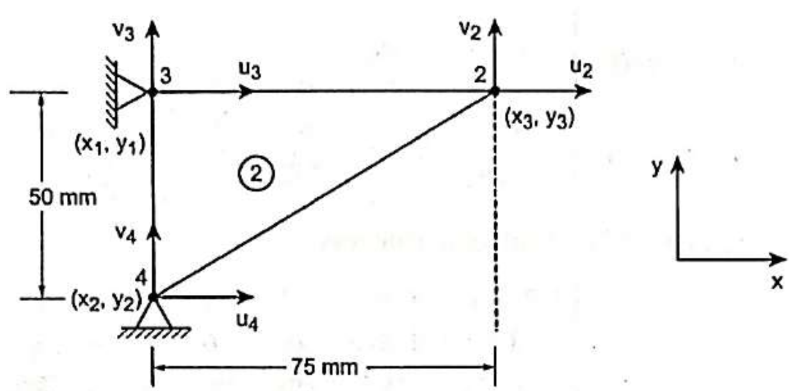


Fig. (iv)

Take node 3 as origin

$$\begin{array}{ll} & x_1 \quad y_1 \\ \text{For node 1:} & (0, \quad 0) \end{array}$$

$$\begin{array}{ll} & x_2 \quad\quad y_2 \\ \text{For node 3:} & (0, \quad -50) \end{array}$$

$$\begin{array}{ll} & x_3 \quad y_3 \\ \text{For node 3:} & (75, \quad 0) \end{array}$$

We know that,

Stiffness matrix [K] = $[B]^T$ [D] [B] A t …(8)

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -50 \\ 1 & 75 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \times 1 \times (0 + (50 \times 75) = \frac{3750}{2}$$

A = 1875 mm^2

Strain – Displacement matrix

$$[B] = \frac{1}{2\text{A}}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \quad \text{... (9)}$$

Where $q_1 = y_2 - y_3 = -50 - 0 = -50$

$$q_2 = y_3 - y_1 = 0 - 0 = 0$$

$$q_3 = y_1 - y_2 = 0 + 50 = 50$$

$$r_1 = x_3 - x_2 = 75 - 0 = 75$$

$$r_2 = x_1 - x_3 = 0 - 75 = -75$$

$$r_3 = x_2 - x_1 = 0 - 0 = 0$$

$$\Rightarrow \quad [B] = \frac{1}{2\text{A}}\begin{bmatrix} -50 & 0 & 0 & 0 & 50 & 0 \\ 0 & 75 & 0 & -75 & 0 & 0 \\ 75 & -50 & -75 & 0 & 0 & 50 \end{bmatrix}$$

Substitute Area, value,

$$\Rightarrow \qquad [B] = \frac{1}{2 \times 1875}\begin{bmatrix} -50 & 0 & 0 & 0 & 50 & 0 \\ 0 & 75 & 0 & -75 & 0 & 0 \\ 75 & -50 & -75 & 0 & 0 & 50 \end{bmatrix}$$

$$[B] = \frac{50}{3750}\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1.5 & 0 & -1.5 & 0 & 0 \\ 1.5 & -1 & -1.5 & 0 & 0 & 1 \end{bmatrix} \qquad \text{... (10)}$$

Stress-Strain relationship matrix [D] for plane stress problem is

$$[D] = \frac{\mathrm{E}}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1 - v}{2} \end{bmatrix} \qquad \text{[From equation no. (3.36)]}$$

$$= \frac{2 \times 10^5}{1 - (0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1 - 0.25}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5 \times}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{2 \times 10^5 \times 0.25}{0.9375}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \qquad \text{... (11)}$$

$$[D][B] = \frac{250 \times 2 \times 10^5 \times 0.25}{0.9375 \times 3750}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1.5 & 0 & -1.5 & 0 & 0 \\ 1.5 & -1 & -1.5 & 0 & 0 & 1 \end{bmatrix}$$

$$= 711.2\begin{bmatrix} -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ -1 & 6 & 0 & -6 & 1 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \qquad \text{... (12)}$$

We know that,

$$[B] = \frac{50}{3750}\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1.5 & 0 & -1.5 & 0 & 0 \\ 1.5 & -1 & -1.5 & 0 & 0 & 1 \end{bmatrix}$$

$$[B]^T = \frac{50}{2 \times 375 \times 10^3}\begin{bmatrix} -1 & 0 & 1.5 \\ 0 & 1.5 & -1 \\ 0 & 0 & -1.5 \\ 0 & -1.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[B]^T[D][B] = \frac{50}{3750} \times 711.2 \begin{bmatrix} -1 & 0 & 1.5 \\ 0 & 1.5 & -1 \\ 0 & 0 & -1.5 \\ 0 & -1.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ -1 & 6 & 0 & -6 & 1 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix}$$

$$= 9.483 \begin{bmatrix} 7.375 & -3.75 & -3.375 & 1.5 & -4 & 2.25 \\ -3.75 & 10.5 & 2.25 & -9 & -1.5 & -1.5 \\ -3.375 & 2.25 & 3.375 & 0 & 0 & -2.25 \\ 1.5 & -9 & 0 & 9 & -1.54 & 0 \\ -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 0 \end{bmatrix}$$

Substitute $[B]^T\ [D][B]$ and A, t values in equation (1),

Stiffness matrix, $[K]_2$

$$= 9.483 \begin{bmatrix} 7.375 & -3.75 & -3.375 & 1.5 & -4 & 2.25 \\ -3.75 & 10.5 & 2.25 & -9 & -1.5 & -1.5 \\ -3.375 & 2.25 & 3.375 & 0 & 0 & -2.25 \\ 1.5 & -9 & 0 & 9 & -1.54 & 0 \\ -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \times 1875$$

$$\times 15$$

$$= 26.67 \times 10^4 \begin{bmatrix} 7.375 & -3.75 & -3.375 & 1.5 & -4 & -2.5 \\ -3.75 & 10.5 & 2.25 & -9 & -1.5 & -1.5 \\ -3.375 & 2.25 & 3.375 & 0 & 0 & -2.25 \\ 1.5 & -9 & 0 & 9 & -1.54 & 0 \\ -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix}$$

$$[K]_2 = 1 \times 10^4 \begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 40.01 & -40.01 \\ -90.01 & 60 & 90.01 & 0 & 0 & -60 \\ 40.01 & -240.03 & 0 & 240.03 & -40.01 & 0 \\ -106.68 & 40.01 & 0 & -40.01 & 106.68 & 0 \\ 60 & -40.01 & -60 & 0 & 0 & 40.01 \end{bmatrix}$$

For element (2), nodal displacements are $u_3 v_3,\ u_4, v_4$ and u_2, v_2 [Refer Fig. (iv)]

$$\text{Stiffness matrix, } [K]_2 = 1\times 10^4 \begin{array}{c} \begin{array}{cccccc} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 40.01 & -40.01 \\ -90.01 & 60 & 90.01 & 0 & 0 & -60 \\ 40.01 & -240.03 & 0 & 240.03 & -40.01 & 0 \\ -106.68 & 40.01 & 0 & -40.01 & 106.68 & 0 \\ 60 & -40.01 & -60 & 0 & 0 & 40.01 \end{bmatrix} \end{array} \begin{array}{c} \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_2 \\ v_3 \end{array}$$

…(13)

[Node: Take, Nodal displacements $u_3 v_3,\ u_4, v_4$ and u_2, v_2 respective to $(x_1 y_1)(x_2 y_2)$ and $(x_3 y_3)$

Global stiffness Matrix [K]

Assemble the stiffness matrix equations (7) and (13),

Global stiffness Matrix [K] =

1×10^4

u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	
196.69	-100.01	-90.01	40.01	0	0	-106.68	60	u_1
-100.01	280.04	60	-240.03	0	0	40.01	-40.01	v_1
-90.01	60	106.68 + 90.01	0 + 0	-106.68	40.01	0 + 0	-40.01 + 60	u_2
40.01	-240.03	0 + 0	40.01 + 240.03	60	-40.01	-60 + -40.01	0 + 0	v_2
0	0	-106.68	60	196.69	-100.01	-90.01	40.01	u_3
0	0	40.01	-40.01	-100.01	280.04	60	-240.03	v_3
-106.68	40.01	0 + 0	-60 + -40.01	-90.01	60	90.01 + 106.68	0 + 0	u_4
60	-40.01	-40.01 -60	0 + 0	40.01	-240.03	0 + 0	240.03 + 40.01	v_4

Global Stiffness matrix, $[K]$ =

$$1\times10^4\begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & 0 & 0 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 0 & 0 & 40.01 & -40.01 \\ -90.01 & 60 & 196.69 & 0 & -106.68 & 40.01 & 0 & -100.01 \\ 40.01 & -240.03 & 0 & 240.04 & 60 & -40.01 & -100.01 & 0 \\ 0 & 0 & -106.68 & 60 & 196.69 & -100.01 & -90.01 & 40.01 \\ 0 & 0 & 40.01 & -40.01 & -100.01 & 280.04 & 60 & -240.03 \\ -106.68 & 40.01 & 0 & -100.01 & -90.01 & 60 & 196.69 & 0 \\ 60 & -40.01 & -100.01 & 0 & 40.01 & -240.03 & 0 & 280.04 \end{bmatrix}$$

We Know that, general force equation is

{ F } = [K] {u }

$$\begin{Bmatrix} F_{1-x} \\ F_{1-y} \\ F_{2-x} \\ F_{2-y} \\ F_{3-x} \\ F_{3-y} \\ F_{4-x} \\ F_{4-y} \end{Bmatrix} = 1\times10^4\begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & 0 & 0 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 0 & 0 & 40.01 & -40.01 \\ -90.01 & 60 & 196.69 & 0 & -106.68 & 40.01 & 0 & -100.01 \\ 40.01 & -240.03 & 0 & 240.04 & 60 & -40.01 & -100.01 & 0 \\ 0 & 0 & -106.68 & 60 & 196.69 & -100.01 & -90.01 & 40.01 \\ 0 & 0 & 40.01 & -40.01 & -100.01 & 280.04 & 60 & -240.03 \\ -106.68 & 40.01 & 0 & -100.01 & -90.01 & 60 & 196.69 & 0 \\ 60 & -40.01 & -100.01 & 0 & 40.01 & -240.03 & 0 & 280.04 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \ldots(14)$$

Applying boundary conditions [Refer Fig. (ii)]

1. Node 3 and Node 4 are fixed. So, u_3, v_3 and u_4, v_4 are zero.

   i.e., $u_3 = v_3 = u_4 = v_4 = 0$.

2. Node 1, is moving in *x* direction. So $u_1 \neq 0$ but, $v_1 = 0$.

3. At node 2, a point load of 100×10^3 N is acting in *(-y)* direction.

   So, $F_{2-y} = -100\times 10^3$ N.

4. Body force is neglected. So, the remaining forces are zero.

   i.e., $F_{1-x} = F_{1-y} = F_{2-x} = F_{2-y} = F_{3-x} = F_{3-y} = F_{4-x} = F_{4-y} = 0$.

Substitute the above values in equation (14),

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -100\times10^3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 1\times10^4\begin{bmatrix} 196.69 & -100.01 & -90.01 & 40.01 & 0 & 0 & -106.68 & 60 \\ -100.01 & 280.04 & 60 & -240.03 & 0 & 0 & 40.01 & -40.01 \\ -90.01 & 60 & 196.69 & 0 & -106.68 & 40.01 & 0 & -100.01 \\ 40.01 & -240.03 & 0 & 240.04 & 60 & -40.01 & -100.01 & 0 \\ 0 & 0 & -106.68 & 60 & 196.69 & -100.01 & -90.01 & 40.01 \\ 0 & 0 & 40.01 & -40.01 & -100.01 & 280.04 & 60 & -240.03 \\ -106.68 & 40.01 & 0 & -100.01 & -90.01 & 60 & 196.69 & 0 \\ 60 & -40.01 & -100.01 & 0 & 40.01 & -240.03 & 0 & 280.04 \end{bmatrix}\begin{Bmatrix} u_1 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation v_1=0, are zero. So delete second row and second column of [K] matrix. Similarly, u_3, v_3, u_4 *and* v_4 are zero. So, delete fifth row fifth column, sixth row sixth column, seventh row seventh column and eight row eight column of [K] matrix. Hence the equation reduces to:

$$\begin{Bmatrix} 0 \\ 0 \\ -100 \times 10^3 \end{Bmatrix} = 1 \times 10^4 \begin{bmatrix} 196.69 & -90.01 & 40.01 \\ -90.01 & 196.69 & 0 \\ 40.01 & 0 & 280.04 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} 0 \\ 0 \\ -10 \end{Bmatrix} = \begin{bmatrix} 196.69 & -90.01 & 40.01 \\ -90.01 & 196.69 & 0 \\ 40.01 & 0 & 280.04 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \ldots(15)$$

$$\Rightarrow \quad \begin{Bmatrix} 0 \\ 0 \\ 49.16 \end{Bmatrix} = \begin{bmatrix} 196.69 & -90.01 & 40.01 \\ 0 & 339.797 & 40.01 \\ 0 & -90.01 & -1336.67 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_1 + 2.1852R_2 \\ R_3 \longrightarrow R_1 - 4.916R_3 \end{matrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ 0 \\ 185.584 \end{Bmatrix} = \begin{bmatrix} 196.69 & -90.01 & 40.01 \\ 0 & 339.797 & 40.01 \\ 0 & -90.01 & -5006.056 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad R_3 \longrightarrow R_1 - 3.775R_3$$

$$\Rightarrow \qquad -5006.056\, v_2 = 185.584$$

$$\Rightarrow \qquad v_2 = -0.0371\, mm$$

$$\Rightarrow \qquad 339.797u_2 + 40.01v_2 = 0$$

$$\Rightarrow \qquad 339.797u_2 + 40.01 \times (-0.0371) = 0$$

$$\Rightarrow \qquad u_2 = 0.00436\, mm$$

$$\Rightarrow \qquad 196.69u_1 - 90.01u_2 + 40.01v_2 = 0$$

$$\Rightarrow \quad 196.69u_1 - 90.01(0.00436) + 40.01(-0.0371) = 0$$

$$\Rightarrow \qquad u_1 = 0.00954\, mm$$

Nodal displacements:

$u_1 = 0.00954mm \qquad v_1 = 0\, mm$

$u_2 = 0.00436mm \qquad v_2 = -0.0371\, mm$

$u_3 = 0 \qquad v_3 = 0$

$u_4 = 0$ $\quad\quad\quad\quad$ $v_4 = 0$

Stress in each element:

We know that, stress, { σ } = [D] [B] { u }

For element (1): [Refer Fig. (iii)].: (Nodal displacements equation u_1, v_1, u_2, v_2 and u_4, v_4)

$$\Rightarrow \text{ stress}, \{\sigma\}_1 = 711.2 \begin{bmatrix} -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ -1 & 6 & 0 & -6 & 1 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

[From equation no. (6)]

$$= 711.2 \begin{bmatrix} -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ -1 & 6 & 0 & -6 & 1 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \times \begin{Bmatrix} 0.00954 \\ 0 \\ 0.00436 \\ -0.0371 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\sigma\}_1 = 711.2 \begin{Bmatrix} (4 \times 0.067) + (1.5 \times -0.0371) \\ 0.00954 + (6 \times -0.0371 \\ (-2.25 \times 0.00954) + (2.25 \times 0.00436 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -12.44 \\ -151.53 \\ -8.289 \end{Bmatrix} N/mm^2$$

⇒ Where, $\quad \sigma_x, \sigma_y$ → Normal stress,

τ_{xy} → Shear stress

For element (2): [Refer Fig. (iv)].: (Nodal displacements equation u_3, v_3, u_4, v_4 and u_2, v_2

Stress, { σ_2 } = $[D]_2$ $[B]_2$ { u }

$$= 711.2 \begin{bmatrix} -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ -1 & 6 & 0 & -6 & 1 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_2 \\ v_2 \end{Bmatrix}$$

[From equation no. (12)]

$$= 711.2 \begin{bmatrix} -4 & 1.5 & 0 & -1.5 & 4 & 0 \\ -1 & 6 & 0 & -6 & 1 & 0 \\ 2.25 & -1.5 & -2.25 & 0 & 0 & 1.5 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.00436 \\ -0.0371 \end{Bmatrix}$$

$$\{\sigma\}_2 = 711.2 \begin{Bmatrix} 4 \times 0.00436 \\ 1 \times 0.00436 \\ 1.5 \times -0.0371 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 12.403 \\ 3.100 \\ -39.578 \end{Bmatrix} N/mm^2$$

Result:

1. Nodal displacements:

   $u_1 = 0.00954, \qquad v_1 = 0,$

   $u_2 = 0.00436mm; \ v_2 = -0.0371,$

   $u_3 = 0 \qquad v_3 = 0\ mm;$

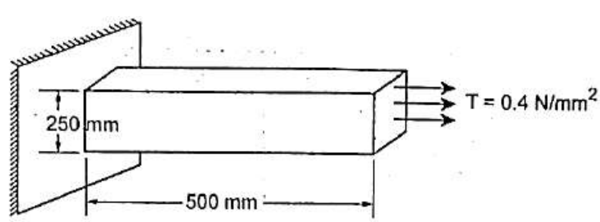
   $u_4 = 0 \qquad v_4 = 0.$

2. Element stresses:

   For element (1) $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -12.44 \\ -151.53 \\ -8.289 \end{Bmatrix} N/mm^2$

   For element (2) $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 12.403 \\ 3.10 \\ -39.578 \end{Bmatrix} N/mm^2$

Tutorial: A thin plate is subjected to surface traction as shown in Fig. (i). Calculate the global stiffness matrix.



Take, t = 25 mm, E = 2 × 10⁵ N/mm² and v = 0.30

Assume plane stress condition.

Given: Thickness, t $= 25$ mm

Young's modulus, E $= 2 \times 10^5$ N/mm^2

Poisson's ratio, v $= 0.30$

Breadth, b $= 250$ mm

Length, l $= 500$ mm

Tensile surface traction, T $= 0.4$ N/mm^2

The tensile surface traction is converted into nodal force.

$$\Rightarrow F = \frac{1}{2}TA = \frac{1}{2} \times T \times (b \times t)$$

$$= \frac{1}{2} \times 0.4 \times 250 \times 25$$

Nodal force, F= 1250 N

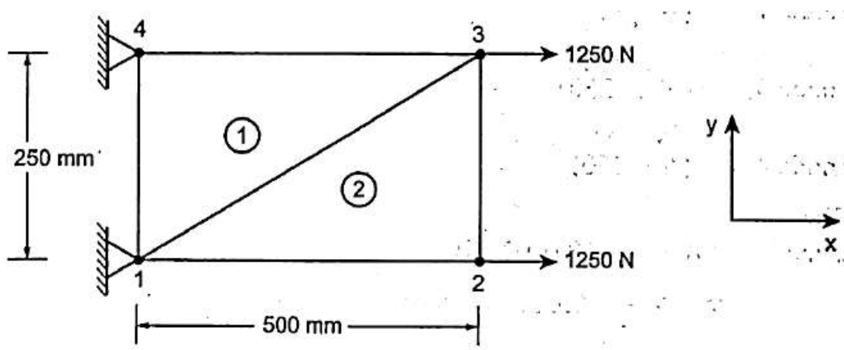


Fig.(ii) Discretized plate

3.12 ELASTICITY EQUATIONS

Elasticity equations are used for solving structural mechanics problems. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained.

There are four basic sets of elasticity equations. They are:

(i) Strain - Displacement relationship equations

(ii) Stress - Strain relationship equations

(iii) Equilibrium equations

(iv) Compatibility equations

3.12.1. Strain-Displacement Relationship Equations

Consider a two dimensional element PQRS as shown in Fig.3.16. When an external force acts on the element, it undergoes deformation and it becomes P'Q'R'S'. Displacement in the x direction is u and y direction is v.

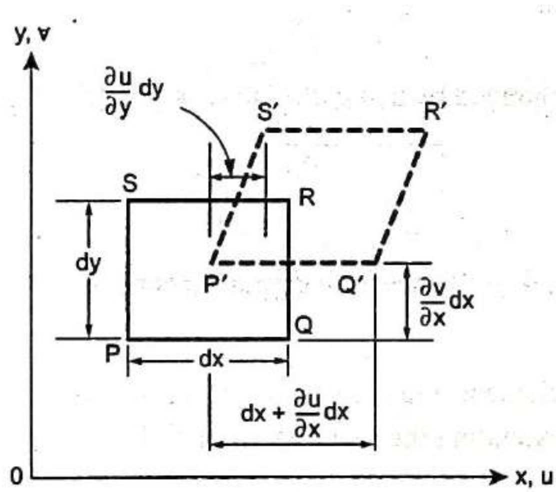


Fig. 3.16. Two dimensional element before and after deformation

We know that, the strain is equal to the ratio of change in length to the original length of the body. Considering the element PQ in *x* direction,

$$\Rightarrow \quad \text{Stain,} \quad e_x = \frac{P'Q' - PQ}{PQ} \quad \text{... (3.40)}$$

We know that, $\quad PQ = dx \quad$ ... (3.41)

and $$(P'Q')^2 = \left[dx + \frac{\partial u}{\partial x} + dx\right]^2 + \left[\frac{\partial v}{\partial x} dx\right]^2$$

calculating $P'Q'$ using the binomial theorem and neglecting the higher order elements i.e., $\left(\frac{\partial u}{\partial x}\right)^2$ and $\left(\frac{\partial v}{\partial x}\right)^2$.

$$\Rightarrow \qquad P'Q' = dx + \frac{\partial u}{\partial x} + dx \qquad \text{... (3.42)}$$

Substituting equations (3.42) and (3.41) in equation (3.40)

$$(3.40) \Rightarrow \qquad e_x = \frac{dx + \frac{\partial u}{\partial x}dx - dx}{dx} \qquad \text{... (3.43)}$$

$$= \frac{dx\left[1 + \frac{\partial u}{\partial x} - 1\right]}{dx} \qquad \text{... (3.44)}$$

$$e_x = \frac{\partial u}{\partial x} \qquad \text{... (3.45)}$$

Similarly, considering the element PS in *y* direction,

$$\Rightarrow \qquad e_y = \frac{\partial v}{\partial y} \qquad \text{... (3.46)}$$

The shear strain γ_{xy} is obtained by using by using the following relation,

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{... (3.47)}$$

Equations (3.45), (3.46) and 3.47) are strain displacement relationships for two dimensional element.

For three dimensional element, the displacement in z-direction is *w*. strain – displacement equations are obtained by extending the two dimensional derivations.

$$\text{Stain in z direction,} \qquad e_z = \frac{\partial w}{\partial z} \qquad \text{... (3.48)}$$

$$\text{Shear Stains are,} \qquad \gamma_{xy} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \text{... (3.49)}$$

$$\gamma_{xy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \text{... (3.50)}$$

Equilibrium equations:

Consider a three dimensional element as shown in Fig. 3.17

It is subjected to normal stresses σ_x, σ_y and σ_z, shear stresses τ_{xy}, τ_{yz} and τ_{zx}, and body forces B_x, B_y and $B_Z$as shown in Fig. 3.17 (b). the stresses acting on the element are assumed to be constant as they act on the width of each face. But they are varying from one face to the opposite. For example, σ_x is acting on the left vertical face, whereas $\sigma_x + \frac{\partial \sigma_x}{\partial x}$ dx is acting on the right vertical face.

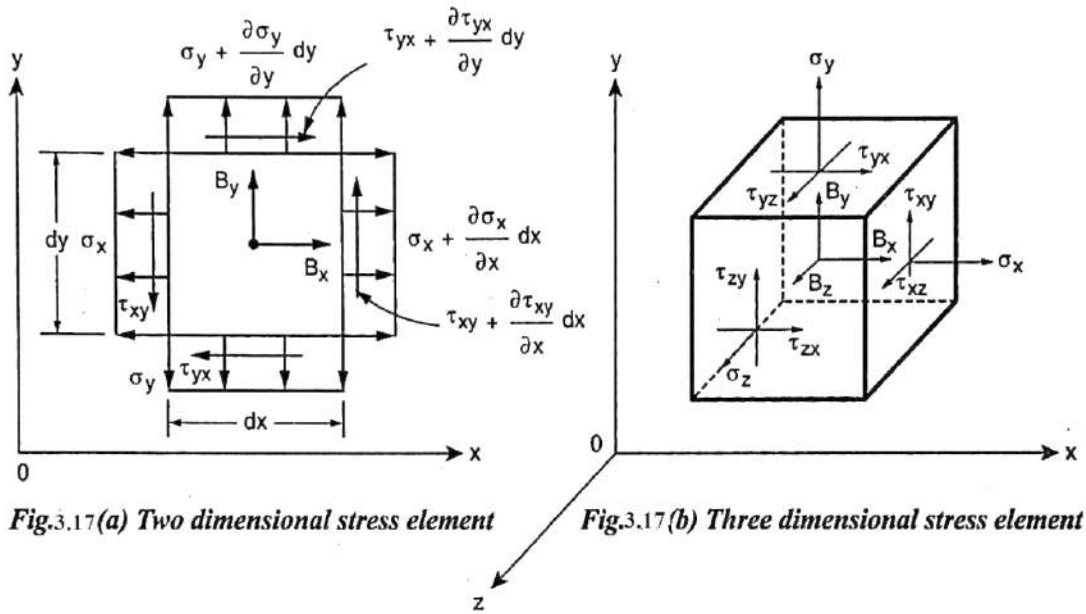


Fig.3.17 (a) Two dimensional stress element ***Fig.3.17 (b) Three dimensional stress element***

Adding all the forces acting on the element in *x* – direction,

$$\Rightarrow \qquad \Sigma F_x = 0$$

$$\Rightarrow \left[\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right] dydz - \sigma_x dydz + \left[\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right] dx\, dz - \tau_{xy}\, dx\, dz$$

$$+ \left[\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz\right] dx\, dy - \tau_{xz}\, dx\, dy + B_x dx\, dy\, dz = 0$$

$$\Rightarrow \sigma_x dydz + \frac{\partial \sigma_x}{\partial x} dxdydz - \sigma_x dydz + \tau_{xy} dx\, dz + \frac{\partial \tau_{xy}}{\partial y} dydx\, dz - \tau_{xy}\, dx\, dz$$

$$+ \tau_{xz} dx\, dy + \frac{\partial \tau_{xz}}{\partial z} dz\, dx\, dy - \tau_{xz}\, dx\, dy + B_x dx\, dy\, dz = 0$$

$$\Rightarrow \quad \frac{\partial \sigma_x}{\partial x} dxdydz + \frac{\partial \tau_{xy}}{\partial y} dx\, dy\, dz + \frac{\partial \tau_{xz}}{\partial z} dx\, dydz + B_x dx\, dy\, dz = 0 \; ... (3.51)$$

Divided by $dxdydz$

$$\Rightarrow \qquad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \qquad \text{... (3.52)}$$

Similarly, adding all the forces acting on the element in the *y* and *z* – directions,

$$\Rightarrow \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + B_y = 0 \qquad \text{... (3.53)}$$

$$\Rightarrow \qquad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + B_z = 0 \qquad \text{... (3.54)}$$

Equations (3.52), (3.53) and (3.54) are equilibrium equations for three dimensional element.

3.13. AXISYMMETRIC ELEMENTS

In previous chapters, we have been concerned with one dimensional elements and two dimensional elements. In this chapter, we consider a special two dimensional element called the axisymmetric element.

Many three dimensional problems in engineering exhibit symmetry about an axis of rotation. Such types of problems are known as axisymmetric problems. These problems can be solved by using two dimensional finite elements. These elements are most conveniently described in cylindrical (r, θ, z) co-ordinates. The required conditions for a problem to be axisymmetric are as follows:

1. The problem domain must be symmetric about the axis of revolution, which is conventionally taken as the z-axis.
2. All boundary conditions must be symmetric about the axis of revolution.
3. All loading conditions must be symmetric about the axis of revolution.

An axisymmetric solid is generated by revolving a plane figure about an axis in the plane.

Finite elements for axisymmetric solids are pictured as triangular element or quadrilateral element as shown in Fig.3.18 and 3.19. But these shapes are actually cross-sections of ring elements.

We begin with the development of the stiffness matrix for the simplest axisymmetric element, the triangular torus, whose vertical cross-section is a plane triangle.

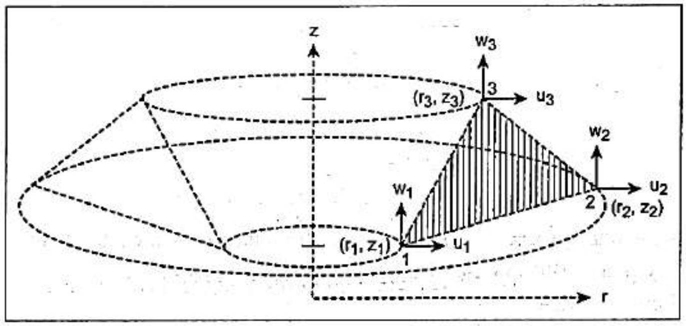


Fig. 3.18. Three-node axisymmetric triangular element

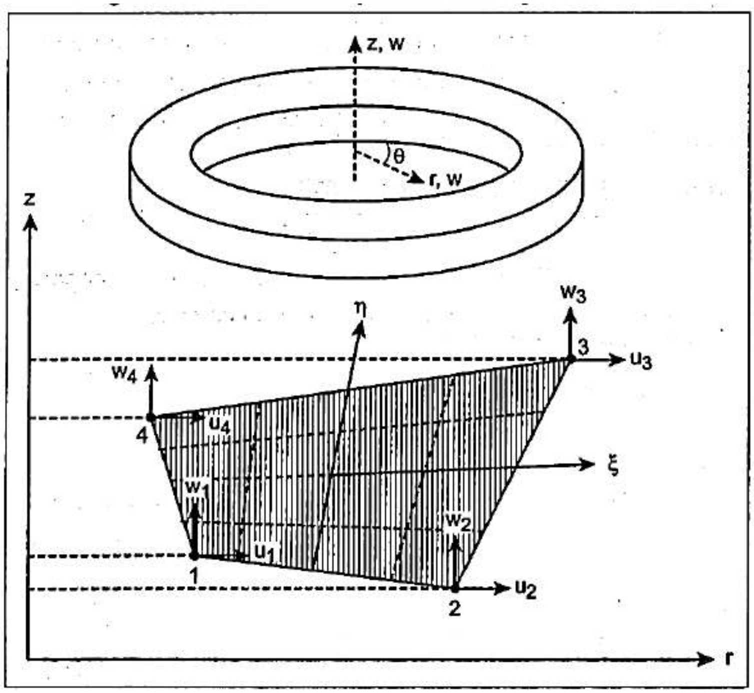


Fig. 3.19 . Four-node axisymmetric quadrilateral element

3.13.1. Axisymmetric Formulation

Consider a typical axisymmetric triangular element with nodes 1, 2 and 3 as shown in Fig.3.20.

In two dimensional problems, the displacements and distributed body force values are indicated by x-y plane. But in case of axisymmetric problems, these values are indicated by r-z plane as shown in Fig.3.20.

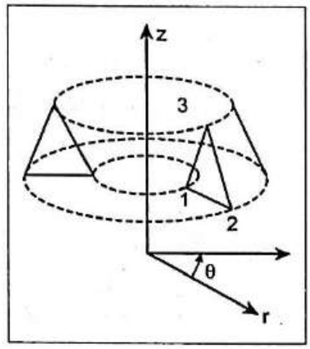


Fig. 3.20. Typical axisymmetric element

For two dimensional problem, the displacement vector u is given by,

$$u(x,y) = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Where, *u* and *v* are the *x* and *y* components of *u* respectively.

In case of axisymmetric problems, the displacement vector *u* is given by,

$$u(r,z) = \begin{Bmatrix} u \\ w \end{Bmatrix}$$

Where, *u* and *w* are *r* and *z* components of *u* respectively.

The stresses and strains for two dimensional element are given by,

$$\text{stress}, \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\text{strain}, \{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

in case of axisymmetric element, stresses and strains are given by,

$$\text{stress}, \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

where, σ_r → Radial stress

σ_z → Longitudinal stress

σ_θ → Circumferential stress

τ_{rz} → Shear stress

$$\text{strain}, \{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

where, e_r → Radial stress

e_z → Longitudinal stress

e_θ → Circumferential stress

γ_{rz} → Shear stress

For two dimensional problem, body force is given by,

$$\mathrm{F} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

In case of axisymmetric problem,

$$\mathrm{F} = \begin{Bmatrix} F_r \\ F_z \end{Bmatrix}$$

3.13.2 Derivation of shape function for Axisymmetric element (Triangular Element)

Consider an axisymmetric triangular element with nodes 1, 2 and 3 as shown in Fig. 3.18 Let the nodal displacements be u_1, w_1, u_2, w_2 and u_3, w_3.

$$\text{Displacement}, \{u\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

Since the triangular element has two degrees of freedom at each node, it has 6 generalized co-ordinates.

Displacement functions, $u = a_1 + a_2 r + a_3 z$ ...(3.55)

$w = a_4 + a_5 r + a_6 z$ ...(3.56)

Where, a_1, a_2, a_3, a_4, a_5 and a_6 are global or generalized co-ordinates.

Let
$$u_1 = a_1 + a_2 r + a_3 z_1$$
$$u_2 = a_1 + a_2 r_2 + a_3 z_2$$
$$u_3 = a_1 + a_2 r_3 + a_3 z_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 0 & r_3 & z_3 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 0 & r_3 & z_3 \end{bmatrix}^{-1}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots(3.57)$$

$$Let, D = \begin{bmatrix} + & - & + \\ 1 & r_1 & z_1 \\ - & + & + \\ 1 & r_2 & z_2 \\ + & - & + \\ 1 & r_3 & z_3 \end{bmatrix}$$

$$D = \frac{C^T}{|D|} \qquad \ldots(3.58)$$

Find the co-factors of matrix D.

$$C_{11} = +\begin{vmatrix} r_2 & z_2 \\ r_3 & z_3 \end{vmatrix} = (r_2 z_3 - r_3 z_2)$$

$$C_{12} = -\begin{vmatrix} 1 & z_2 \\ 1 & z_3 \end{vmatrix} = -(z_3 - z_2) = (z_2 - z_3)$$

$$C_{13} = +\begin{vmatrix} 1 & r_2 \\ 1 & r_3 \end{vmatrix} = +(r_3 - z_2)$$

$$C_{21} = -\begin{vmatrix} r_1 & z_1 \\ r_2 & z_3 \end{vmatrix} = (r_1 z_3 - r_3 z_1) = r_3 z_3 - r_1 z_3$$

$$C_{22} = +\begin{vmatrix} 1 & z_1 \\ 1 & z_3 \end{vmatrix} = (z_3 - z_1)$$

$$C_{23} = -\begin{vmatrix} 1 & r_1 \\ 1 & r_3 \end{vmatrix} = -(r_3 - r_1) = (r_1 - r_3)$$

$$C_{31} = +\begin{vmatrix} r_1 & r_2 \\ r_2 & z_2 \end{vmatrix} = r_1 z_2 - r_2 z_1$$

$$C_{32} = -\begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} = -(z_2 - z_1) = (z_1 - z_2)$$

$$C_{33} = +\begin{vmatrix} 1 & r_1 \\ 1 & r_2 \end{vmatrix} = (r_2 - z_1)$$

$$\Rightarrow C = \begin{bmatrix} (r_2 z_3 - r_3 z_2) & (z_2 - z_3) & r_3 - z_2 \\ (r_3 z_1 - r_1 z_3) & z_3 - z_1 & r_1 - r_3 \\ r_1 z_2 - r_2 z_1 & z_1 - z_2 & r_2 - z_1 \end{bmatrix}$$

$$\mathrm{C}^{\mathrm{T}} = \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - z_2 & r_1 - r_3 & r_2 - z_1 \end{bmatrix} \quad \text{... (3.59)}$$

We know that,

$$D = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 0 & r_3 & z_3 \end{bmatrix}$$

$$|D| = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 0 & r_3 & z_3 \end{bmatrix}$$

$$|D| = 1(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2) \quad \text{...(3.60)}$$

Substitute C^{T} and D values in equation (3.58),

$$(4.20) \Rightarrow D^{-1} = \frac{1}{(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2)} \times \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - z_2 & r_1 - r_3 & r_2 - z_1 \end{bmatrix}$$

Substitute D^{-1} value in equation (3.57),

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 0 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2)}$$

$$\times \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - z_2 & r_1 - r_3 & r_2 - z_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{... (3.61)}$$

The area of the triangle can be expressed as a function of the r, z co-ordinates of the nodes 1, 2 and 3.

$$A = \frac{1}{2}\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 0 & r_3 & z_3 \end{bmatrix}$$

$$A = \frac{1}{2} 1(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2)$$

$$\Rightarrow 2A = (r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2) \qquad ...(3.62)$$

Substitute equation (3.62) in equation (3.61),

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2\text{A}}\begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - z_2 & r_1 - r_3 & r_2 - z_1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \ldots (3.63)$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2\text{A}}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (3.64)$$

Where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\alpha_2 = r_3 z_1 - r_1 z_3$; $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$ $\beta_2 = z_3 - z_1$ $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - z_2$ $\gamma_2 = r_1 - r_3$ $\gamma_3 = r_2 - z_1$

From equation (3.55), we know that,

$$u = a_1 + a_2 r + a_3 z$$

We can write this equation in matrix form,

$$u = [1 \ \ r \ \ z]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= [1 \ \ r \ \ z] \times \frac{1}{2\text{A}}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{[From equation(3.64)]}$$

$$= \frac{1}{2\text{A}}[1 \ \ r \ \ z]\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2\mathrm{A}}[\alpha_1 + \beta_1 r + \gamma_1 z \quad \alpha_2 + \beta_2 r + \gamma_2 z \quad \alpha_3 + \beta_3 r + \gamma_3 z] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

[Note: (1 × 3) × (3 × 3) = (1 × 3)]

$$\Rightarrow u = \left[\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A} \quad \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A} \quad \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}\right] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

$$u = [N_1 \;\; N_2 \;\; N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad ..(3.65)$$

Similarly, $$w = [N_1 \;\; N_2 \;\; N_3] \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \qquad ..(3.66)$$

$$\text{Where, Shape function, } N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$$

$$N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

We can write equations (3.65) and (3.66) as follows:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \qquad \ldots(3.67)$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3 \qquad \ldots(3.68)$$

Assembling the equations (3.67) and (3.68) in matrix form,

Displacement function,

$$u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \qquad \ldots(3.69)$$

3.13.3 Strain – Displacement Matrix [B] for Axisymmetric Triangular Element

Displacement function for axisymmetric triangular element is given by,

Displacement function,

$$u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

or

we can write, $$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \qquad \text{...(3.70)}$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3 \qquad \text{...(3.71)}$$

The strain components are,

Radial strain, $e_r = \dfrac{\partial u}{\partial r} = \dfrac{\partial N_1}{\partial r} u_1 + \dfrac{\partial N_2}{\partial r} u_2 + \dfrac{\partial N_2}{\partial r} u_3$

$$\Rightarrow e_r = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_2}{\partial r} u_3 \qquad \text{... (3.72)}$$

Circumferential strain, $e_\theta = \dfrac{u}{r}$

$$\Rightarrow e_r = \frac{N_1}{r} u_1 + \frac{N_2}{r} u_2 + \frac{N_2}{r} u_3 \qquad \text{... (3.73)}$$

Longitudinal strain, $e_z = \dfrac{\partial w}{\partial z}$

$$\Rightarrow e_z = \frac{\partial N_1}{\partial z} w_1 + \frac{\partial N_2}{\partial z} w_2 + \frac{\partial N_2}{\partial z} w_3 \qquad \text{... (3.74)}$$

Shear strain, $\gamma_{rz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}$

$$\gamma_{rz} = \frac{\partial N_1}{\partial z} u_1 + \frac{\partial N_2}{\partial z} u_2 + \frac{\partial N_3}{\partial z} u_3 + \frac{\partial N_1}{\partial r} w_1 + \frac{\partial N_2}{\partial r} w_2 + \frac{\partial N_3}{\partial r} w_3 \qquad \text{... (3.75)}$$

Arranging equations (3.72), (3.73), (3.74) and (3.75) in matrix form,

$$\Rightarrow \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial r} & 0 & \dfrac{\partial N_2}{\partial r} & 0 & \dfrac{\partial N_3}{\partial r} & 0 \\ \dfrac{N_1}{r} & 0 & \dfrac{N_2}{r} & 0 & \dfrac{N_3}{r} & 0 \\ 0 & \dfrac{\partial N_1}{\partial z} & 0 & \dfrac{\partial N_2}{\partial z} & 0 & \dfrac{\partial N_3}{\partial z} \\ \dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_1}{\partial r} & \dfrac{\partial N_2}{\partial z} & \dfrac{\partial N_2}{\partial r} & \dfrac{\partial N_3}{\partial z} & \dfrac{\partial N_3}{\partial r} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \text{... (3.76)}$$

From equation (3.65) or (3.66), we know that,

Shape function, $N_1 = \dfrac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$

$$N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

Partial differentiation $\Rightarrow \dfrac{\partial N_1}{\partial r} = \dfrac{\beta_1}{2A}$

$$\frac{\partial N_2}{\partial r} = \frac{\beta_2}{2A}$$

$$\frac{\partial N_3}{\partial r} = \frac{\beta_3}{2A}$$

$$\frac{N_1}{r} = \frac{1}{2A}\left(\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}\right)$$

$$\frac{N_2}{r} = \frac{1}{2A}\left(\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}\right)$$

$$\frac{N_3}{r} = \frac{1}{2A}\left(\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}\right)$$

$$\frac{\partial N_1}{\partial z} = \frac{\gamma_1}{2A}$$

$$\frac{\partial N_2}{\partial z} = \frac{\gamma_2}{2A}$$

$$\frac{\partial N_3}{\partial z} = \frac{\gamma_3}{2A}$$

Substitute $\frac{\partial N_1}{\partial r}, \frac{\partial N_2}{\partial r}, \frac{\partial N_3}{\partial r}, \frac{N_1}{r}, \frac{N_2}{r}, \frac{N_3}{r}, \frac{\partial N_1}{\partial z}, \frac{\partial N_2}{\partial z}$, and $\frac{\partial N_3}{\partial z}$ values in Equation (3.76).

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

The above equation is in the form of,

$$\{ e \} = [B] \{ u \}$$

[B] = Strain – Displacement matrix

$$= \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad ..(3.77)$$

Where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\quad \alpha_2 = r_3 z_1 - r_1 z_3$; $\quad \alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3 \quad \beta_2 = z_3 - z_1 \quad \beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - z_2 \quad \gamma_2 = r_1 - r_3 \quad \gamma_3 = r_2 - z_1$

3.13.4 Stress – Strain Relationship Matrix [D] for Axisymmetric Triangular Element

By using Hooke's law, we derived the following normal stresses equations.

$$\sigma_x = \frac{E}{(1+v)(1-2v)} \left[e_x(1-v) + v e_y + v e_z \right]$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)} \left[v e_x(1-v) + e_y + v e_z \right]$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)} \left[v e_x + v e_y + (1-v) e_z \right]$$

$$\tau_{xy} = \frac{E}{(1+v)(1-2v)}\left(\frac{1-2v}{2}\right) \times \gamma_{xz}$$

Substitute x = r and y = θ in the above equations,

$\Rightarrow$ Radial stress, $$\sigma_r = \frac{E}{(1+v)(1-2v)}[e_r(1-v) + ve_\theta + ve_z] \quad \text{... (3.78)}$$

Circumferential stress, $$\sigma_\theta = \frac{E}{(1+v)(1-2v)}[ve_r(1-v) + e_\theta + ve_z] \quad \text{... (3.79)}$$

Longitudinal strain, $$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[ve_x + ve_y + (1-v)e_z\right] \quad \text{... (3.80)}$$

Shear strain, $$\tau_{rz} = \frac{E}{(1+v)(1-2v)}\left(\frac{1-2v}{2}\right) \times \gamma_{rz} \quad \text{... (3.81)}$$

Arranging the above equations, (3.78), (3.79), (3.80) and (3.81) in matrix form,

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} \quad \text{... (3.82)}$$

The above equation is in the form of,

{ σ } = [D] { E }

Where [D] = Stress - Strain relationship matrix

$$= \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \quad \text{... (3.83)}$$

3.13.5 Assemblage of the Element stiffness Matrix [K]

We know that,

$$\text{Stiffness matrix, } [K] = \int_v [B]^T[D][B]dV = [B]^T[D][B]\int_v dV$$

$$= [B]^T[D][B]V$$

$$\text{Stiffness matrix, } [K] = 2\pi r\, A\, [B]^T [D][B] \qquad \ldots(3.84)$$

$[\because V = 2\pi r A]$

Where,

$$\text{Co} - \text{ordinate}, r = \frac{r_1 + r_2 + r_3}{3}$$

A = Area of the triangular element = ½ (b × h)

[B] = Strain – Displacement matrix

$$= \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\quad \alpha_2 = r_3 z_1 - r_1 z_3$; $\quad \alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3 \quad \beta_2 = z_3 - z_1 \quad \beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - z_2 \quad \gamma_2 = r_1 - r_3 \quad \gamma_3 = r_2 - z_1$

and $$\text{Co} - \text{ordinate}, r = \frac{r_1 + r_2 + r_3}{3}$$

[D] = Stress- Strain relationship matrix

$$= \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

Where, E → Young's modulus

v → Poisson's ratio

3.14. SOLVED PROBLEMS -AXISYMMETRIC ELEMENT

Example 3.10

The nodal co-ordinates for an axisymmetric triangular element are given below

$r_1 = 10$ mm ; $z_1 = 10$ mm

$r_2 = 30$ mm ; $z_2 = 10$ mm

$r_3 = 30$ mm ; $z_3 = 40$ mm

Evaluate [B] Matrix for that element

Given:

$r_1 = 10$ mm ; $z_1 = 10$ mm

$r_2 = 30$ mm ; $z_2 = 10$ mm

$r_3 = 30$ mm ; $z_3 = 40$ mm

To find:

Strain displacement matrix [B]

Solution: We know that,

Strain - Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad \text{... (1)}$$

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}$$

$$= \frac{1}{2}(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2)$$

$$= \frac{1}{2}[(30 \times 40) - (30 \times 10) - 10(40 - 10) + 10(30 - 30)]$$

$$= \frac{1}{2} \times 600$$

$A = 300 \text{ mm}^2$

$$\text{Co} - \text{ordinate,} \qquad r = \frac{r_1 + r_2 + r_3}{3} = \frac{10 + 30 + 30}{3}$$

r = 23.334 mm

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 10 + 40}{3}$$

z = 20 mm

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (30 \times 40) - (30 \times 10)$$

$$\boldsymbol{\alpha_1 = 900\, mm^2}$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (30 \times 10) - (10 \times 40)$$

$$\boldsymbol{\alpha_2 = -100\, mm^2}$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (10 \times 10) - (30 \times 10)$$

$$\boldsymbol{\alpha_3 = -200\, mm^2}$$

$$\beta_1 = z_2 - z_3 = 10 - 40$$

$$\boldsymbol{\beta_1 = -30\, mm}$$

$$\beta_2 = z_3 - z_1 = 40 - 10$$

$$\boldsymbol{\beta_2 = 30\, mm}$$

$$\beta_3 = z_1 - z_2 = 10 - 10$$

$$\boldsymbol{\beta_3 = 0}$$

$$\gamma_1 = r_3 - z_2 = 30 - 30$$

$$\boldsymbol{\gamma_1 = 0}$$

$$\gamma_2 = r_1 - r_3 = 10 - 30$$

$$\boldsymbol{\gamma_2 = -20\, mm}$$

$$\gamma_3 = r_2 - z_1 = 30 - 10$$

$$\gamma_3 = 20\, mm$$

$$\Rightarrow \ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{900}{23.334} + (-30) + \frac{0 \times 20}{23.334} = 8.571\, mm$$

$$\Rightarrow \ \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{-100}{23.334} + (30) + \frac{(-20 \times 20)}{23.334} = 8.571\, mm$$

$$\Rightarrow \ \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-200}{23.334} + 0 + \frac{20 \times 20}{23.334} = 8.571\, mm$$

Substitute, A, $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2$ and γ_3 values in equation (1),

$$(1) \Rightarrow [B] = \frac{1}{2 \times 300}\begin{bmatrix} -30 & 0 & 30 & 0 & 0 & 0 \\ 8.571 & 0 & 8.571 & 0 & 8.571 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -30 & -20 & 30 & 20 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0.0142 & 0 & 0.0142 & 0 & 0.0142 & 0 \\ 0 & 0 & 0 & -0.0333 & 0 & 0.0333 \\ 0 & -0.05 & -0.0333 & 0.05 & 0.0333 & 0 \end{bmatrix}$$

Result: Strain-Displacement matrix

$$[B] = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0.0142 & 0 & 0.0142 & 0 & 0.0142 & 0 \\ 0 & 0 & 0 & -0.0333 & 0 & 0.0333 \\ 0 & -0.05 & -0.0333 & 0.05 & 0.0333 & 0 \end{bmatrix}$$

Example 3.11

The nodal co-ordinates for an axisymmetric triangular element are given below

$r_1 = 20$ mm ; $z_1 = 40$ mm

$r_2 = 40$ mm ; $z_2 = 40$ mm

$r_3 = 30$ mm ; $z_3 = 60$ mm

Evaluate [B] Matrix for that element

Given: Coordinates

$r_1 = 20$ mm ; $z_1 = 40$ mm

$r_2 = 40$ mm ; $z_2 = 40$ mm

$r_3 = 30$ mm ; $z_3 = 60$ mm

To find: Strain displacement matrix [B]

Solution:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \ldots (1)$$

[From equation no. (4.39)]

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \qquad \text{[From equation no. (4.24)]}$$

$$= \frac{1}{2}(r_2 z_3 - r_3 z_2) - r_1(z_3 - z_2) + z_1(r_3 - r_2)$$

$$= \frac{1}{2}[(40 \times 60) - (30 \times 40) - 20(60 - 40) + 40(30 - 40)]$$

$$= \frac{1}{2}[1200 - 400 - 400]$$

A = 200 mm^2

$$\text{Co} - \text{ordinate,} \qquad r = \frac{r_1 + r_2 + r_3}{3} = \frac{20 + 40 + 30}{3}$$

r = 30 mm

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{40 + 40 + 60}{3}$$

z = 46.667 mm

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (40 \times 60) - (30 \times 40)$$

$$\boldsymbol{\alpha_1 = 1200\ mm^2}$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (30 \times 40) - (20 \times 60)$$

$$\boldsymbol{\alpha_2 = 0}$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (20 \times 40) - (40 \times 40)$$

$$\boldsymbol{\alpha_3 = -800\ mm^2}$$

$$\beta_1 = z_2 - z_3 = 40 - 60$$

$$\boldsymbol{\beta_1 = -20\ mm}$$

$$\beta_2 = z_3 - z_1 = 60 - 40$$

$$\boldsymbol{\beta_2 = 20\ mm}$$

$$\beta_3 = z_1 - z_2 = 40 - 40$$

$$\boldsymbol{\beta_3 = 0}$$

$$\gamma_1 = r_3 - z_2 = 30 - 40$$

$$\boldsymbol{\gamma_1 = -10\ mm}$$

$$\gamma_2 = r_1 - r_3 = 20 - 30$$

$$\boldsymbol{\gamma_2 = -10\ mm}$$

$$\gamma_3 = r_2 - z_1 = 40 - 20$$

$$\gamma_3 = 20\ mm$$

$$\Rightarrow \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{1200}{30} + (-20) + \frac{-10 \times 46.667}{30} = 40 - 20 - 15.556$$

$$= 4.444\ mm$$

$$\Rightarrow \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{0}{30} + (20) + \frac{(-10 \times 46.667)}{30} = 4.444\ mm$$

$$\Rightarrow \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-800}{30} + 0 + \frac{20 \times 46.667}{30} \qquad = 4.444\ mm$$

Substitute, A, $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2$ and γ_3 values in equation (1),

$$(1) \Rightarrow [B] = \frac{1}{2 \times 200}\begin{bmatrix} -20 & 0 & 20 & 0 & 0 & 0 \\ 4.444 & 0 & 4.444 & 0 & 4.444 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -20 & -10 & 20 & 20 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0.0111 & 0 & 0.0111 & 0 & 0.0111 & 0 \\ 0 & -0.025 & 0 & -0.025 & 0 & 0.05 \\ -0.025 & -0.05 & -0.025 & 0.05 & 0.05 & 0 \end{bmatrix}$$

Result: Strain-Displacement matrix

$$[B] = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0.0111 & 0 & 0.0111 & 0 & 0.0111 & 0 \\ 0 & -0.025 & 0 & -0.025 & 0 & 0.05 \\ -0.025 & -0.05 & -0.025 & 0.05 & 0.05 & 0 \end{bmatrix}$$

Example 3.12

For the element shown in fig (i), determine the stiffness matrix. Take E= 200 Gpa and v = 0.25. The coordinated shown in fig (i) are in millimetres.

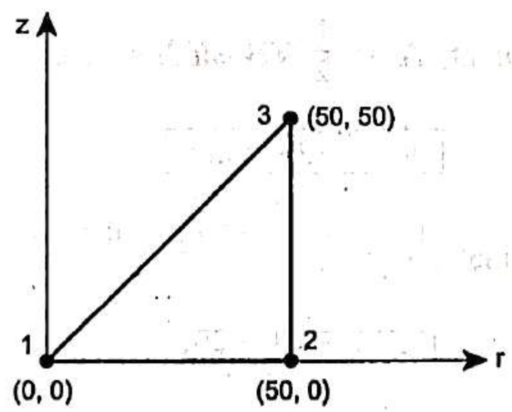


Fig. (i)

Given:

$r_1 = 0$ mm ; $z_1 = 0$ mm

$r_2 = 50$ mm ; $z_2 = 0$ mm

$r_3 = 50$ mm ; $z_3 = 50$ mm

Young's Modulus, E = 200 GPa

$= 2 \times 10^5$ N/mm^2

Poisson's ratio, $v = 0.25$

To find:

Element stiffness matrix [K].

Solution: For axisymmetric triangular element, stiffness matrix [K] is given by,

$$[K] = 2\pi r\, A\, [B]^T [D][B] \qquad \ldots(1)$$

Where, A = Area of the triangular element

$$= \frac{1}{2}\begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}$$

We can calculate as,

Area of the triangular element,

$$\text{A} = \frac{1}{2} \times \text{Breadth} \times \text{Height} = \frac{1}{2} \times 50 \times 50$$

$$\text{A} = 1250\, mm^2 \qquad \ldots(2)$$

$$\text{Co} - \text{ordinate}, \qquad r = \frac{r_1 + r_2 + r_3}{3} = \frac{0 + 50 + 50}{3}$$

r = 33.333 mm …(3)

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 50}{3}$$

z = 16.666 mm …(4)

we know that,

Stress - Strain relationship matrix

$$= \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

$$\Rightarrow [D] = \frac{2 \times 10^5}{(1+0.25)(1-(2 \times 0.25))}\begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \frac{1-2(0.25)}{2} \end{bmatrix}$$

$$= 320 \times 10^3 \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix} = 320 \times 10^3 \times 0.25 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D] = 80 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We know that, Strain - Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad \text{...(6)}$$

Where,

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (50 \times 50) - (50 \times 0)$$

$$\boldsymbol{\alpha_1 = 2500\ mm^2}$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (50 \times 0) - (0 \times 50)$$

$$\boldsymbol{\alpha_2 = 0}$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (0 \times 0) - (50 \times 0)$$

$$\boldsymbol{\alpha_3 = 0}$$

$$\beta_1 = z_2 - z_3 = 0 - 50$$

$$\boldsymbol{\beta_1 = -50\ mm}$$

$$\beta_2 = z_3 - z_1 = 50 - 0$$

$$\boldsymbol{\beta_2 = 50\ mm}$$

$$\beta_3 = z_1 - z_2 = 0 - 0$$

$$\boldsymbol{\beta_3 = 0}$$

$$\gamma_1 = r_3 - z_2 = 50 - 50$$

$$\boldsymbol{\gamma_1 = 0}$$

$$\gamma_2 = r_1 - r_3 = 0 - 50$$

$$\boldsymbol{\gamma_2 = -\ 50\ mm}$$

$$\gamma_3 = r_2 - z_1 = 50 - 0$$

$$\gamma_3 = 50\ mm$$

$$\Rightarrow \ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{2500}{23.334} + (-50) + 0 = 25\ mm$$

$$\Rightarrow \ \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = 0 + (30) + \frac{(-20 \times 20)}{23.334} = 25\ mm$$

$$\Rightarrow \ \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{50 \times 16.666}{33.333} = 25\ mm$$

Substitute, A, $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2$ and γ_3 values in equation (6),

$$[B] = \frac{1}{2A}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & 0 & 0 & -50 & 0 & 50 \\ 0 & -50 & -50 & 50 & 50 & 0 \end{bmatrix}$$

Substitute Area A, value

$$[B] = \frac{1}{2 \times 1250}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & 0 & 0 & -50 & 0 & 50 \\ 0 & -50 & -50 & 50 & 50 & 0 \end{bmatrix} \qquad \text{... (7)}$$

$$\Rightarrow [D][B] = 80 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \frac{1}{2 \times 1250}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & 0 & 0 & -50 & 0 & 50 \\ 0 & -50 & -50 & 50 & 50 & 0 \end{bmatrix}$$

$$= 80 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \frac{1}{2 \times 1250}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$= 800 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$= 800\begin{bmatrix} -6+1+0+0 & 0+0+0+0 & 6+1+0+0 & 0+0-2+0 & 0+1+0+0 & 0+1+2+0 \\ -2+3+0+0 & 0+0+0+0 & 2+3+0+0 & 0+0-2+0 & 0+3+0+0 & 0+0+2+0 \\ -2+1+0+0 & 0+0+0+0 & 2+1+0+0 & 0+0-6+0 & 0+1+0+0 & 0+0+6+0 \\ 0+0+0+0 & 0+0+0-2 & 0+0+0-2 & 0+0+0+2 & 0+0+0+2 & 0+0+0+0 \end{bmatrix}$$

$$[D][B] = 800\begin{bmatrix} -5 & 0 & 7 & -2 & 1 & 2 \\ 1 & 0 & 5 & -2 & 3 & 2 \\ -1 & 0 & 3 & -6 & 1 & 6 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix} \qquad \ldots(8)$$

We know that,

$$[B] = \frac{1}{2\times 1250}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & 0 & 0 & -50 & 0 & 50 \\ 0 & -50 & -50 & 50 & 50 & 0 \end{bmatrix}$$

$$[B] = \frac{25}{2\times 1250}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$[B]^T = 0.01\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \ldots(9)$$

$$[B]^T[D][B] = 0.01\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \times 800\begin{bmatrix} -5 & 0 & 7 & -2 & 1 & 2 \\ 1 & 0 & 5 & -2 & 3 & 2 \\ -1 & 0 & 3 & -6 & 1 & 6 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$= 0.01\times 800\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} -5 & 0 & 7 & -2 & 1 & 2 \\ 1 & 0 & 5 & -2 & 3 & 2 \\ -1 & 0 & 3 & -6 & 1 & 6 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$= 800\begin{bmatrix} 10+1+0+0 & 0+0+0+0 & -14+5+0+0 & 4-2+0+0 & -2+3+0+0 & -4+2+0+0 \\ 0+0+0+0 & 0+0+0+4 & 0+0+0+4 & 0+0+0-4 & 0+0+0-4 & 0+0+0+0 \\ -10+1+0+0 & 0+0+0+4 & 14+5+0+0 & -4-2+0-0 & 2+3+0-4 & 4+2+0+0 \\ 0+0+2+0 & 0+0+0-4 & 0+0-6-4 & 0+0+12+4 & 0+0+0+2 & 0+0-12+0 \\ 0+1+0+0 & 0+0+0-4 & 0+5+0-4 & 0-2+0+4 & 0+3+0+4 & 0+2+0+0 \\ 0+0-2+0 & 0+0+0+0 & 0+0+6+0 & 0+0-12+0 & 0+0+2+0 & 0+0+12+0 \end{bmatrix}$$

$$[B]^T[D][B] = 8\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix}$$

Substitute $[B]^T[D][B]$ value in equation (1),

$$\Rightarrow \qquad [K] = 2\pi r A \times 8\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix}$$

$$= 2 \times \pi \times 33.333 \times 1250 \times 8\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix}$$

$$\text{Stiffness matrix } [K] = 2.094 \times 10^6\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix}$$

It may be noted that stiffness matrix [K] is symmetric.

Result:

$$\text{Stiffness matrix } [K] = 2.094 \times 10^6\begin{bmatrix} 11 & 0 & -9 & 2 & 1 & -2 \\ 0 & 4 & 4 & -4 & -4 & 0 \\ -9 & 4 & 23 & -10 & 1 & 6 \\ 2 & -4 & -10 & 16 & 2 & -12 \\ 1 & -4 & 1 & 2 & 7 & 2 \\ -2 & 0 & 6 & -12 & 2 & 12 \end{bmatrix} N/mm$$

Example 3.13

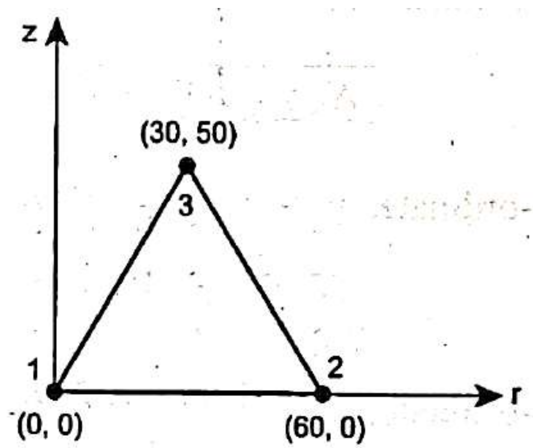
For the axisymmetric element shown in Fig. (i), determine the element stresses. Take E = 2.1×10^5 N/mm² and v=0.25.

The co-ordinates shown in Fig. (i) are in millimeters. The nodal displacements are:

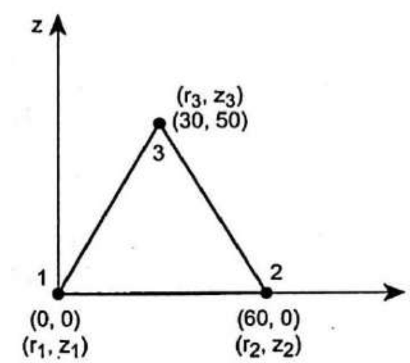
$u_1 = 0.05$ mm ; $w_1 = 0.03$ mm

$u_2 = 0.02$ mm ; $w_2 = 0.02$ mm

$u_3 = 0$ mm ; $w_3 = 0$ mm



Given:



$r_1 = 0$ mm ; $z_1 = 0$ mm

$r_2 = 60$ mm ; $z_2 = 0$ mm

$r_3 = 30$ mm ; $z_3 = 50$ mm

Nodal displacements:

$u_1 = 0.05$ mm ; $w_1 = 0.03$ mm

$u_2 = 0.02$ mm ; $w_2 = 0.02$ mm

$u_3 = 0$ mm ; $w_3 = 0$ mm

Young's modulus, $E = 2.1 \times 10^5$ N/mm^2

Poisson's ratio, $v = 0.25$

To find: Element stresses:

(i) Radial stress, σ_r

(ii) Circumferential stress, σ_θ

(iii) Longitudinal stress, σ_z

(iv) Shear stress, τ_{rz}

Solution: we know that,

Stress { σ } = [D] [B] { u }

$$\Rightarrow \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = [D][B] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \ldots(1)$$

Area of the triangular element,

$$\mathrm{A} = \frac{1}{2} \times \text{Breadth} \times \text{Height} = \frac{1}{2} \times 60 \times 50$$

$$\mathrm{A} = 1500\, mm^2 \quad \ldots(2)$$

$$\text{Co}-\text{ordinate,} \qquad r = \frac{r_1 + r_2 + r_3}{3} = \frac{0 + 60 + 30}{3}$$

r = 30 mm …(3)

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 50}{3}$$

z = 16.667 mm …(4)

we know that,

Stress - Strain relationship matrix

$$= \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix}$$

$$\Rightarrow [D] = \frac{2.1 \times 10^5}{(1+0.25)(1-(2 \times 0.25))} \begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \dfrac{1-2(0.25)}{2} \end{bmatrix}$$

$$= 336 \times 10^3 \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix} = 336 \times 10^3 \times 0.25 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D] = 80 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots(5)$$

We know that, Strain - Displacement matrix

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad \ldots(6)$$

Where,

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (60 \times 50) - (30 \times 0)$$

$$\boldsymbol{\alpha_1 = 3000\ mm^2}$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (30 \times 0) - (0 \times 50)$$

$$\boldsymbol{\alpha_2 = 0}$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (0 \times 0) - (60 \times 0)$$

$$\boldsymbol{\alpha_3 = 0}$$

$$\beta_1 = z_2 - z_3 = 0 - 50$$

$$\boldsymbol{\beta_1 = -50\ mm}$$

$$\beta_2 = z_3 - z_1 = 50 - 0$$

$$\boldsymbol{\beta_2 = 50\ mm}$$

$$\beta_3 = z_1 - z_2 = 0 - 0$$

$$\boldsymbol{\beta_3 = 0}$$

$$\gamma_1 = r_3 - z_2 = 30 - 60$$

$$\boldsymbol{\gamma_1 = -30\ mm}$$

$$\gamma_2 = r_1 - r_3 = 0 - 30$$

$$\boldsymbol{\gamma_2 = -30\ mm}$$

$$\gamma_3 = r_2 - z_1 = 60 - 0$$

$$\gamma_3 = 60\ mm$$

$$\Rightarrow \quad \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{3000}{23.334} + (-50) + \frac{(-30 \times 16.667)}{30} = 33.33\,mm$$

$$\Rightarrow \quad \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = 0 + 50 + \frac{(-30 \times 16.667)}{30} = 33.333\,mm$$

$$\Rightarrow \quad \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{60 \times 16.667}{30} = 33.33\,mm$$

Substitute, A, $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2$ and γ_3 values in equation (6),

$$[B] = \frac{1}{2 \times 1500}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$[B] = 3.3333 \times 10^{-4}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \quad \text{... (7)}$$

$$\Rightarrow [D][B] = 80 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times 3.3333$$

$$\times 10^{-4}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$= 28\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$[D][B] = 28\begin{bmatrix} -116.67 & -30 & 183.33 & -30 & 33.33 & 60 \\ 49.99 & -30 & 149.99 & -30 & 99.99 & 60 \\ -16.67 & -90 & 83.33 & -90 & 33.33 & 180 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \quad \text{...(8)}$$

Substitute $[B]^T[D][B]$ value u_1, w_1, u_2, w_2 and u_3, w_3 values in equation (1), equation (1) becomes,

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = 28 \begin{bmatrix} -116.67 & -30 & 183.33 & -30 & 33.33 & 60 \\ 49.99 & -30 & 149.99 & -30 & 99.99 & 60 \\ -16.67 & -90 & 83.33 & -90 & 33.33 & 180 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = 28 \begin{Bmatrix} -3.666 \\ 4 \\ -3.666 \\ -2.6 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = 28 \begin{Bmatrix} -102.65 \\ 112 \\ -102.65 \\ -72.8 \end{Bmatrix}$$

⇒ Radial stress, σ_r = -102.65 N/mm^2

Circumferential stress, σ_θ = 112 N/mm^2

Longitudinal stress, σ_z = -102.65 N/mm^2

Shear stress, τ_{rz} = -72.8 N/mm^2

Result: Element stresses:

Radial stress, σ_r = -102.65 N/mm^2

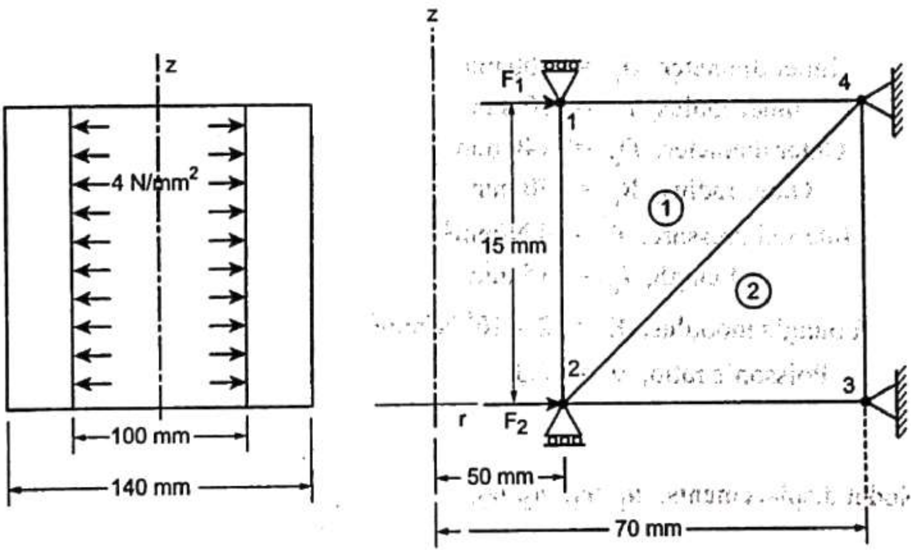
Circumferential stress, σ_θ = 112 N/mm^2

Longitudinal stress, σ_z = -102.65 N/mm^2

Shear stress, τ_{rz} = -72.8 N/mm^2

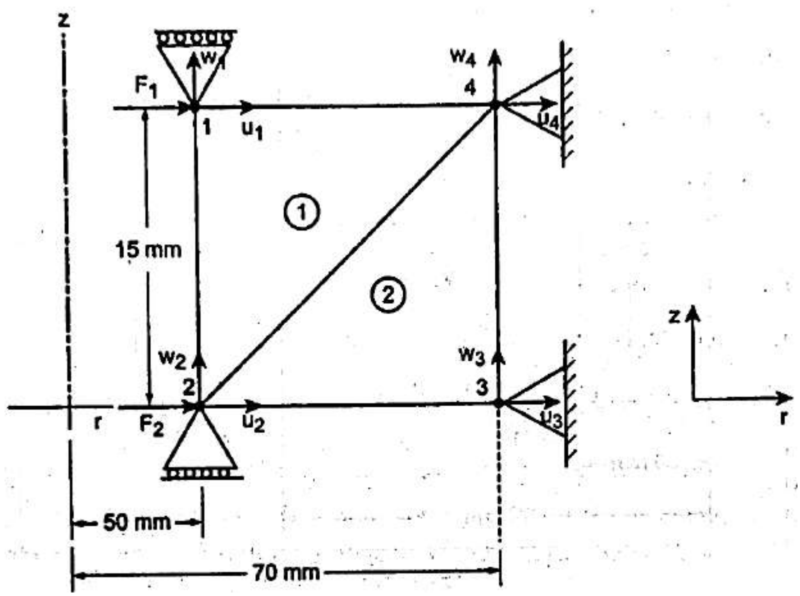
Example 3.14:

A long hollow cylinder of inside diameter 100 mm and outside diameter 140 mm is subjected to an internal pressure of 4 N/mm^2 as shown in Fig. (i). By using two elements on the 15 mm length shown in Fig. (i), calculate the displacements at the inner radius.



Take $E = 2 \times 10^5$ N/mm^2 and $v = 0.3$.

Given:



Fig(ii)

Inner diameter,	$d_e = 100$ mm
Inner radius,	$r_e = 50$ mm
Outer diameter,	$D_e = 140$ mm

Outer radious, $R_e = 70$ mm

Internal Pressure, $P = 4$ N/mm^2

Length , $l_e = 15$ mm

Young's modulus, $E = 2.1 \times 10^5$ N/mm^2

Poisson's ratio, $v = 0.3$

To find: Nodal displacements: $u_1, w_1, u_2, w_2, u_3, w_3, u_4, w_4$

Solution: For element (1) (Nodal displacements: $u_1, w_1, u_2, w_2, u_3, w_3, u_4, w_4$)

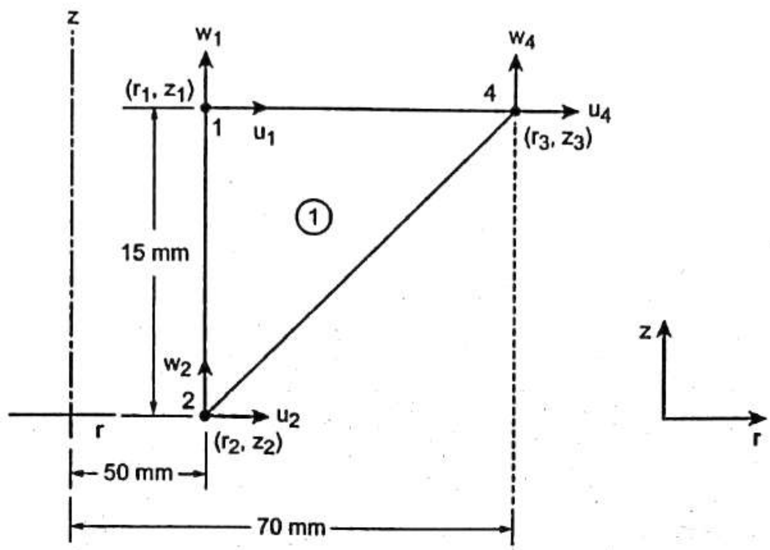


Fig. (iii)

Co-ordinates:

At node 1: $r_1 = 50$ mm ;

$z_1 = 15$ mm

At node 2: $r_2 = 50$ mm ;

$z_2 = 0$ mm

At node 4: $r_3 = 70$ mm ;

$z_3 = 15$ mm

We know that, $$r = \frac{r_1 + r_2 + r_3}{3} = \frac{50 + 50 + 70}{3}$$

$$\mathbf{r = 56.6667\ mm} \quad \ldots(1)$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{15 + 0 + 15}{3}$$

$$\mathbf{z = 10\ mm} \quad \ldots(2)$$

Area of the triangular element,

$$\text{A} = \frac{1}{2} \times \text{Breadth} \times \text{Height} = \frac{1}{2} \times (70 - 50) \times 15$$

$$\text{A} = \frac{1}{2} \times 20 \times 15$$

$$\text{A} = 150\ mm \quad \ldots(3)$$

we know that,

stiffness for axisymmetric triangular element,

$$[K] = 2\pi r\ A\ [B]^T [D][B] \quad \ldots(4)$$

Stress - Strain relationship matrix

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

$$\Rightarrow [D] = \frac{2 \times 10^5}{(1+0.3)(1-(2 \times 0.3))} \begin{bmatrix} 1-0.3 & 0.3 & 0.3 & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.25 & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-2(0.3)}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{0.5} \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$[D] = 3846.6153 \times 10^3 \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

We know that, Strain - Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad ...(5)$$

Where,

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (50 \times 15) - (70 \times 0)$$

$$\boldsymbol{\alpha_1 = 750\ mm^2}$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (70 \times 15) - (50 \times 15)$$

$$\boldsymbol{\alpha_2 = 300\ mm^2}$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (50 \times 0) - (50 \times 15)$$

$$\boldsymbol{\alpha_3 = -750\ mm^2}$$

$$\beta_1 = z_2 - z_3 = 0 - 15$$

$$\boldsymbol{\beta_1 = -15\ mm}$$

$$\beta_2 = z_3 - z_1 = 15 - 15$$

$$\boldsymbol{\beta_2 = 0}$$

$$\beta_3 = z_1 - z_2 = 15 - 0$$

$$\boldsymbol{\beta_3 = 15\ mm}$$

$$\gamma_1 = r_3 - z_2 = 70 - 50$$

$$\boldsymbol{\gamma_1 = 20\ mm}$$

$$\gamma_2 = r_1 - r_3 = 50 - 70$$

$$\boldsymbol{\gamma_2 = -20\ mm}$$

$$\gamma_3 = r_2 - z_1 = 50 - 50$$

$$\gamma_3 = 0\ mm$$

$$\Rightarrow \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{750}{56.6667} + (-15) + \frac{20 \times 10}{56.6667} = 1.7647\ mm$$

$$\Rightarrow \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{300}{56.6667} + 0 + \frac{-20 \times 10}{56.6667} = 1.7647\ mm$$

$$\Rightarrow \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-750}{56.6667} + 15 + 0 = 1.7647\ mm$$

Substitute, A, $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2$ and γ_3 values in equation (6),

$$[B] = \frac{1}{2 \times 150} \begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[B] = 3.3333 \times 10^{-4} \begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix} \ldots (6)$$

$$\Rightarrow [D][B] = 384.6153 \times 10^3 \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \times 3.3333$$

$$\times 10^{-4} \begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix}$$

$$[D][B] = 1.282 \times 10^3 \begin{bmatrix} -9.9706 & 6 & 0.5294 & -6 & 11.0294 & 0 \\ -3.2647 & 6 & 1.2353 & -6 & 5.7353 & 0 \\ -3.9706 & 14 & 0.5294 & -14 & 5.0294 & 0 \\ 4 & -3 & -4 & 0 & 0 & 3 \end{bmatrix}$$

We know that,

$$[B] = 3.3333 \times 10^{-3} \begin{bmatrix} -15 & 0 & 0 & 0 & 15 & 0 \\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 \\ 20 & -15 & -20 & 0 & 0 & 15 \end{bmatrix} \ldots (6)$$

$$[B]^T = 3.3333 \times 10^{-3} \begin{bmatrix} -15 & 1.7647 & 0 & 20 \\ 0 & 0 & 20 & -15 \\ 0 & 1.7647 & 0 & -20 \\ 0 & 0 & -20 & 0 \\ 15 & 1.7647 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$[B]^T[D][B] = 3.333 \times 10^{-3} \begin{bmatrix} -15 & 1.7647 & 0 & 20 \\ 0 & 0 & 20 & -15 \\ 0 & 1.7647 & 0 & -20 \\ 0 & 0 & -20 & 0 \\ 15 & 1.7647 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix} \times 1.28$$

$$\times 10^{-3} \begin{bmatrix} -9.9706 & 6 & 0.5294 & -6 & 11.0294 & 0 \\ -3.2647 & 6 & 1.2353 & -6 & 5.7353 & 0 \\ -3.9706 & 14 & 0.5294 & -14 & 5.0294 & 0 \\ 4 & -3 & -4 & 0 & 0 & 3 \end{bmatrix}$$

$$[B]^T[D][B] = 4.2733 \begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.412 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & -100.588 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.1211 & -60 \\ 79.412 & -280 & -10.588 & 280 & -100.588 & 0 \\ -155.3202 & -100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

Substitute $[B]^T[D][B]$ value in equation (1),

$[K]_1 = 2 \times \pi \times 56.6667 \times 150 \times 4.2733 \times$

$$\begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.412 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & -100.588 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.1211 & -60 \\ 79.412 & -280 & -10.588 & 280 & -100.588 & 0 \\ -155.3202 & -100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

$[K]_1$

$$= 228.2246 \times 10^3 \begin{bmatrix} 223.7978 & -139.4118 & -85.7611 & 79.412 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & -100.588 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.1211 & -60 \\ 79.412 & -280 & -10.588 & 280 & -100.588 & 0 \\ -155.3202 & -100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

$$[K]_1 = 10^6 \begin{array}{c} \begin{array}{cccccc} u_1 & w_1 & u_2 & w_2 & u_4 & w_4 \end{array} \\ \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & -35.448 & 13.693 \\ -31.817 & 74.173 & 16.110 & -63.903 & 22.957 & -10.270 \\ -19.573 & 16.110 & 18.755 & -2.416 & 2.310 & -13.693 \\ 18.124 & -63.903 & -2.416 & 63.903 & -22.957 & 0 \\ -35.448 & 22.957 & 2.310 & -22.957 & 40.068 & 0 \\ 13.693 & -10.270 & -13.693 & 0 & 0 & 10.270 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \ldots (7)$$

For element (1) (Nodal displacements: $u_2, w_2, u_3, w_3, u_4, w_4$)

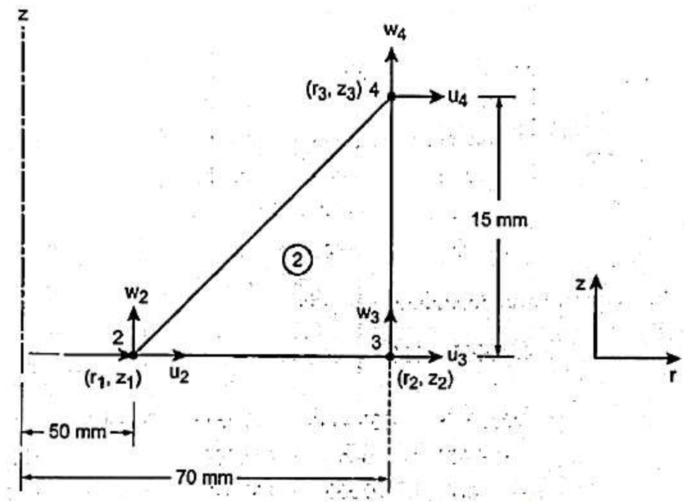


Fig. (iv)

Co-ordinates:

At node 2: $r_1 = 50$ mm ;

$z_1 = 0$ mm

At node 3: $r_2 = 70$ mm ;

$z_2 = 0$ mm

At node 4: $r_3 = 70$ mm ;

$z_3 = 15$ mm

We know that, $$r = \frac{r_1 + r_2 + r_3}{3} = \frac{50 + 70 + 70}{3}$$

r = 63.333 mm

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 15}{3}$$

z = 5 mm

Area of the triangle,

$$\mathrm{A} = \frac{1}{2} \times \text{Breadth} \times \text{Height} = \frac{1}{2} \times (70 - 50) \times 15$$

$$\mathrm{A} = 150\, mm$$

we know that,

stiffness for axisymmetric triangular element,

$$[K] = 2\pi r\, A\, [B]^T [D][B] \qquad \text{...(8)}$$

Stress - Strain relationship matrix

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix}$$

$$\Rightarrow [D] = \frac{2 \times 10^5}{(1+0.3)(1-(2 \times 0.3))} \begin{bmatrix} 1-0.3 & 0.3 & 0.3 & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.25 & 1-0.3 & 0 \\ 0 & 0 & 0 & \dfrac{1-2(0.3)}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{0.5} \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$[D] = 384..6153 \times 10^3 \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \qquad \text{...(9)}$$

We know that, Strain - Displacement matrix

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad \text{... (10)}$$

Where,

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (70 \times 15) - (70 \times 0)$$

$$\boldsymbol{\alpha_1 = 1050\ mm^2}$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (70 \times 0) - (50 \times 15)$$

$$\boldsymbol{\alpha_2 = -750\ mm^2}$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (50 \times 0) - (70 \times 15)$$

$$\boldsymbol{\alpha_3 = 0}$$

$$\beta_1 = z_2 - z_3 = 0 - 15$$

$$\boldsymbol{\beta_1 = -15\ mm}$$

$$\beta_2 = z_3 - z_1 = 15 - 0$$

$$\boldsymbol{\beta_2 = 15\ mm}$$

$$\beta_3 = z_1 - z_2 = 0 - 0$$

$$\boldsymbol{\beta_3 = 0}$$

$$\gamma_1 = r_3 - z_2 = 70 - 70$$

$$\boldsymbol{\gamma_1 = 0}$$

$$\gamma_2 = r_1 - r_3 = 50 - 70$$

$$\boldsymbol{\gamma_2 = -20\ mm}$$

$$\gamma_3 = r_2 - z_1 = 70 - 50$$

$$\gamma_3 = 20\ mm$$

$$\Rightarrow\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} = \frac{1050}{63.3333} + (-15) + 0 = 1.579\ mm$$

$$\Rightarrow\ \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{-750}{63.3333} + 15 + \frac{-20 \times 5}{63.3333} = 1.579\ mm$$

$$\Rightarrow\ \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{20 \times 5}{63.3333} = 1.579\ mm$$

Substitute, A, $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2$ and γ_3 values in equation (10), we get

$$[B] = \frac{1}{2 \times 150}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$[B] = 3.3333 \times 10^{-3}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$\Rightarrow [D][B] = 384.6153 \times 10^3 \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \times 3.3333$$

$$\times 10^{-3}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$[D][B] = 1.282 \times 10^3 \begin{bmatrix} -10.0263 & 0 & 10.9737 & -6 & 0.4737 & 6 \\ -3.3947 & 0 & 5.6053 & -6 & 1.1053 & 6 \\ -4.0263 & 0 & 4.9737 & -14 & 0.4737 & 14 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

We know that,

$$[B] = 3.3333 \times 10^{-3}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 20 & -15 & -20 & 15 & 20 & 15 \end{bmatrix}$$

$$[B]^T = 3.3333 \times 10^{-3}\begin{bmatrix} -15 & 1.579 & 0 & 0 \\ 0 & 0 & 0 & -15 \\ 15 & 1.579 & 0 & -20 \\ 0 & 0 & -20 & 15 \\ 0 & 1.579 & 0 & 20 \\ 0 & 0 & 20 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 3.333 \times 10^{-3}\begin{bmatrix} -15 & 1.579 & 0 & 0 \\ 0 & 0 & 0 & -15 \\ 15 & 1.579 & 0 & -20 \\ 0 & 0 & -20 & 15 \\ 0 & 1.579 & 0 & 20 \\ 0 & 0 & 20 & 0 \end{bmatrix} \times 1.28$$

$$\times 10^{-3}\begin{bmatrix} -10.0263 & 0 & 10.9737 & -6 & 0.4737 & 6 \\ -3.3947 & 0 & 5.6053 & -6 & 1.1053 & 6 \\ -4.0263 & 0 & 4.9737 & -14 & 0.4737 & 14 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 4.2733\begin{bmatrix} 145.034 & 0 & -155.755 & 80.526 & -5.360 & -80.526 \\ 0 & 45 & 60 & -45 & -160 & 0 \\ -155.755 & 60 & 253.456 & -159.474 & -71.149 & 99.474 \\ 80.526 & -45 & -159.474 & 325 & 50.526 & -280 \\ -5.360 & -60 & -71.149 & 50.526 & 81.745 & 9.474 \\ -80.526 & 0 & 99.474 & -280 & 9.474 & 280 \end{bmatrix}$$

Substitute $[B]^T[D][B]$ value in equation (8), we get,

$[K]_2 = 2 \times \pi \times 63.3333 \times 150 \times 4.2733 \times$

$$\begin{bmatrix} 145.034 & 0 & -155.755 & 80.526 & -5.360 & -80.526 \\ 0 & 45 & 60 & -45 & -160 & 0 \\ -155.755 & 60 & 253.456 & -159.474 & -71.149 & 99.474 \\ 80.526 & -45 & -159.474 & 325 & 50.526 & -280 \\ -5.360 & -60 & -71.149 & 50.526 & 81.745 & 9.474 \\ -80.526 & 0 & 99.474 & -280 & 9.474 & 280 \end{bmatrix}$$

$[K]_2 = 255.074$

$$\times 10^3\begin{bmatrix} 145.034 & 0 & -155.755 & 80.526 & -5.360 & -80.526 \\ 0 & 45 & 60 & -45 & -160 & 0 \\ -155.755 & 60 & 253.456 & -159.474 & -71.149 & 99.474 \\ 80.526 & -45 & -159.474 & 325 & 50.526 & -280 \\ -5.360 & -60 & -71.149 & 50.526 & 81.745 & 9.474 \\ -80.526 & 0 & 99.474 & -280 & 9.474 & 280 \end{bmatrix}$$

$$
[K]_2 = 10^6
\begin{array}{c}
\begin{array}{cccccc} u_2 & w_2 & u_3 & w_3 & u_4 & w_4 \end{array} \\
\begin{bmatrix}
36.9994 & 0 & -39.729 & 20.540 & -1.367 & -20.540 \\
0 & 11.478 & 15.304 & -11.478 & -15.304 & 0 \\
-39.729 & 15.304 & 64.650 & -40.678 & -18.148 & 25.373 \\
20.540 & -11.478 & -40.678 & 82.899 & 12.887 & -71.421 \\
-1.367 & -15.304 & -18.148 & 12.887 & 20.851 & 2.417 \\
20.540 & 0 & 25.373 & -71.421 & 2.417 & 71.421
\end{bmatrix}
\end{array}
\begin{Bmatrix} u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{Bmatrix}
\quad \dots (11)
$$

Assemble the equation (7) and (11),

Global stiffness Matrix [K] =

$10^6 \times$

	u_1	w_1	u_2	w_2	u_3	w_3	u_4	w_4
u_1	51.076 + 0	-31.817 + 0	-19.573 + 0	18.124 + 0	0	0	-35.448 + 0	13.639 + 0
w_1	-31.817 + 0	74.173 + 0	16.110 + 0	-63.903 + 0	0	0	22.957 + 0	-10.270 + 0
u_2	-19.573 + 0	16.110 + 0	18.755 + 36.994	-2.416 + 0	-39.729 + 0	20.540 + 0	2.310 - 1.367	-13.693 - 20.540
w_2	18.124 + 0	-63.903 + 0	-2.416 + 0	63.903 + 11.478	0 + 15.304	0 - 11.478	-22.957 - 15.304	0 + 0
u_3	0	0	0 + -39.729	0 + 15.304	0 + 64.650	0 + -40.678	0 + -18.148	0 + 25.373
w_3	0	0	20.540 + 0	-11.478 + 0	-40.678 + 0	82.899 + 0	12.887 + 0	-71.421 + 0

-35.448 + 0	22.957 + 0	2.310 - 1.367	-22.957 - 15.304	0 + - 18.148	0 + 12.887	40.068 + 20.851	0 + 2.417	u_4
13.693 + 0	-10.270 + 0	- 13.693 - 20.540	0 + 0	0 + 25.373	0 + -71.421	0 + 2.417	10.270 + 71.421	w_4

[Combining equation (7)+(11)]

Global stiffness Matrix [K] =

	u_1	w_1	u_2	w_2	u_3	w_3	u_4	w_4	
	51.076	-31.817	-19.573	18.124	0	0	-35.448	13.639	u_1
	-31.817	74.173	16.110	-63.903	0	0	22.957	-10.270	w_1
	-19.573	16.110	55.749	-2.416	-39.729	20.540	0.943	-34.233	u_2
$10^6 \times$	18.124	-63.903	-2.416	75.381	15.304	-11.478	-38.261	0	w_2
	0	0	-39.729	15.304	64.650	-40.678	-18.148	25.373	u_3
	0	0	20.540	-11.478	-40.678	82.899	12.887	-71.421	w_3
	-35.448	22.957	0.943	-38.261	-18.148	12.887	60.919	2.417	u_4
	13.693	-10.270	-34.233	0	25.373	-71.421	2.417	81.691	w_4

We know that, { F } = [K] { u }

$$\begin{Bmatrix} F_{1u} \\ F_{1y} \\ F_{2u} \\ F_{2w} \\ F_{3u} \\ F_{3w} \\ F_{4u} \\ F_{4w} \end{Bmatrix} = 10^6 \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & 0 & 0 & -35.448 & 13.6390 \\ -31.817 & 74.173 & 16.110 & -63.903 & 0 & 0 & 22.957 & -10.270 \\ -19.573 & 16.110 & 55.749 & -2.416 & -39.729 & 20.540 & 0.943 & -34.233 \\ 18.124 & -63.903 & -2.416 & 75.381 & 15.304 & -11.478 & -38.261 & 0 \\ 0 & 0 & -39.729 & 15.304 & 64.650 & -40.678 & -18.148 & 25.373 \\ 0 & 0 & 20.540 & -11.478 & -40.678 & 82.899 & 12.887 & -71.421 \\ -35.448 & 22.957 & 0.943 & -38.261 & -18.148 & 12.887 & 60.919 & 2.417 \\ 13.693 & -10.270 & -34.233 & 0 & 25.373 & -71.421 & 2.417 & 81.691 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{Bmatrix}$$

Forces: we know that,

$$\text{Forces, } F_{1u} = F_{2u} = \frac{2\pi\, r_e \times l_e \times P}{2} = \frac{2 \times \pi \times 50 \times 15 \times 4}{2}$$

$$F_{1u} = F_{2u} = 9424.77\ N$$

The remaining forces are zero. i.e., $F_{1w}, F_{2w}, F_{3u}, F_{3w}, F_{4u}$ and F_{4w} are zero.

Displacements [Refer Fig. (ii)]

1. Node 1 is moving in r direction. So. $u_1 \neq 0$, but $w_1 = 0$.
2. Node 2 is moving in r direction. So. $u_2 \neq 0$, but $w_2 = 0$.
3. Node 3 and 4 are fized. So. u_3, w_3, u_4 and w_4 are zero.

Substitute nodal forces and nodal displacements values in equation (12),

$$(12) \Rightarrow \begin{Bmatrix} 9424.77 \\ 0 \\ 9424.77 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 10^6 \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & 0 & 0 & -35.448 & 13.6390 \\ -31.817 & 74.173 & 16.110 & -63.903 & 0 & 0 & 22.957 & -10.270 \\ -19.573 & 16.110 & 55.749 & -2.416 & -39.729 & 20.540 & 0.943 & -34.233 \\ 18.124 & -63.903 & -2.416 & 75.381 & 15.304 & -11.478 & -38.261 & 0 \\ 0 & 0 & -39.729 & 15.304 & 64.650 & -40.678 & -18.148 & 25.373 \\ 0 & 0 & 20.540 & -11.478 & -40.678 & 82.899 & 12.887 & -71.421 \\ -35.448 & 22.957 & 0.943 & -38.261 & -18.148 & 12.887 & 60.919 & 2.417 \\ 13.693 & -10.270 & -34.233 & 0 & 25.373 & -71.421 & 2.417 & 81.691 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Delete second row, second column, fourth row, fourth column, fifth row, fifth column, sixth row, sixth column, seventh row, seventh column and eight row and eight column of the above matrix. hence the equation reduces to.

$$\begin{Bmatrix} 9424.77 \\ 9424.77 \end{Bmatrix} = 10^6 \begin{bmatrix} 51.076 & -19.573 \\ -19.573 & 55.749 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$9424.77 = 10^6(51.076\, u_1 - 19.573 u_2) \qquad \ldots(13)$$

$$9424.77 = 10^6(-19.573\, u_1 + 55.749 u_2) \qquad \ldots(14)$$

$$Eqn. (13) \times 19.573 \Rightarrow 184.47 \times 10^3 = 10^6(999.710\, u_1 - 383.102 u_2)$$

$$Eqn. (14) \times 51.076 \Rightarrow 481.379 \times 10^3 = 10^6(-999.710\, u_1 + 2847.4352 u_2)$$

$$\text{Solving, } 665.849 \times 10^3 = 10^6(2464.333) u_2$$

$$\Rightarrow \qquad u_2 = 2.70 \times 10^{-4} mm$$

Substitute u_2 value in equation (13) or (14)

$$\Rightarrow 9424.77 = 10^6(51.076\, u_1 - 19.573 \times 2.70 \times 10^{-4} mm)$$

$$\Rightarrow \quad u_1 = 2.88 \times 10^{-4} mm$$

Result: Displacements:

$u_1 = 2.88 \times 10^{-4} mm$ $\quad w_1 = 0$

$u_2 = 2.70 \times 10^{-4} mm$ $\quad w_2 = 0$

$u_3 = 0$ $\quad w_3 = 0$

$u_4 = 0$ $\quad w_4 = 0$

TWO MARKS QUESTIONS & ANSWERS

1. How do you define two dimensional elements?

Two dimensional elements are defined by three or more nodes in a two dimensional plane (i.e., x, y plane). The basic element useful for two dimensional analysis is the triangular element.

2. What is CST element?

Three noded triangular element is known Constant Strain Triangle (CST), which is shown in Fig.(i). It has six unknown displacement degrees of freedom (u_1 v_1 , u_2 v_2 , u_3 v_3). The element is called CST because it has a constant strain throughout it.

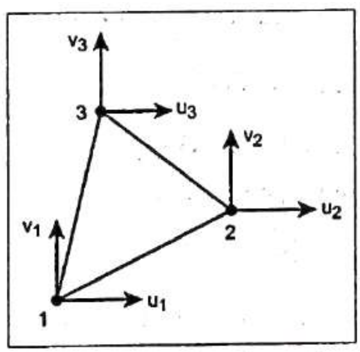


Fig. (i). Constant strain triangular element

3. What is LST element?

Six noded triangular element is known as Linear Strain Triangle (LST), which is shown in Fig.(ii). It has twelve unknown displacement degrees of freedom. The displacement functions for the element are quadratic instead of linear as in the CST.

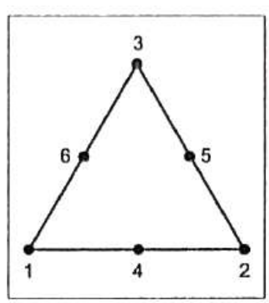


Fig. (ii). Linear strain triangular element

4. What is QST element?

Ten noded triangular element is known as Quadratic Strain Triangle (QST) which is shown in Fig.(iii). It is also called cubic displacement triangle.

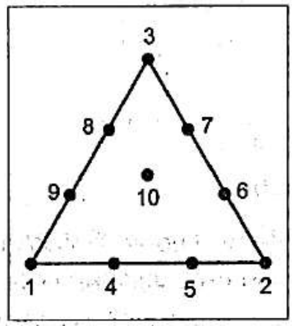


Fig. (iii). Quadratic strain triangle element

5. What is meant by plane stress analysis?

Plane stress is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.

6. Define plane strain analysis.

Plane strain is defined to be a state of strain in which the strain normal to the xy plane and the shear strains are assumed to be zero.

7. Write a displacement function equation for CST element.

$$\text{Displacement function, u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Where, N_1, N_2, N_3 are shape functions.

8. Write a strain-displacement matrix for CST element.

Strain-Displacement matrix for CST element is,

$$[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

Where, A = Area of the element

$$q_1 = y_2 - y_3; \qquad q_2 = y_3 - y_1; \qquad q_3 = y_1 - y_2$$

$$r_1 = x_3 - x_2; \qquad r_2 = x_1 - x_3; \qquad r_3 = x_2 - x_1$$

9. Write down the stress-strain relationship matrix for plane stress condition.

For plane stress problems, stress-strain relationship matrix is,

$$[D] = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}$$

Where, E = Young's modulus

v = Poisson's ratio

10. Write down the stress-strain relationship matrix for plane strain condition.

For plane strain problems, Stress-strain relationship matrix is,

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix}$$

11. Write down the stiffness matrix equation for two dimensional CST element.

Stiffness matrix [K] = $[B]^T$ [D] [B] A t ...(1)

Where, [B] → Strain – Displacement matrix

[D] → Stress-strain matrix

A → Area of the element

t → Thickness of the element

12. Write down the expression for the shape functions for a constant strain triangular element.

For CST element,

Shape Function, $$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

where, $p_1 = x_2 y_3 - x_3 y_2$; $p_2 = x_3 y_1 - x_1 y_3$; $p_3 = x_1 y_2 - x_2 y_1$

$q_1 = y_2 - y_3$; $q_3 = y_1 - y_2$; $q_2 = y_3 - y_1$

$r_1 = x_3 - x_2$; $r_2 = x_1 - x_3$; $r_3 = x_3 - x_1$

13. What is axisymmetric element?

Many three dimensional problems in engineering exhibit symmetry about an axis of rotation. Such types of problems are solved by a special two dimensional element called as axisymmetric element.

14. What are the conditions for a problem to be axisymmetric?

1. The problem domain must be symmetric about the axis of revolution.
2. All boundary conditions must be symmetric about the axis of revolution.
3. All loading conditions must be symmetric about the axis of revolution.

15. Write down the displacement equation for an axisymmetric triangular element.

Displacement function,

$$u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

16. **Write down the shape functions for an axisymmetric triangular element**.

$$\text{Where, Shape function, } N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$$

$$N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

Where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\alpha_2 = r_3 z_1 - r_1 z_3$; $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$ $\beta_2 = z_3 - z_1$ $\beta_3 = z_1 - z_2$

$$\gamma_1 = r_3 - z_2 \qquad \gamma_2 = r_1 - r_3 \qquad \gamma_3 = r_2 - z_1$$

17. Give the Strain-Displacement matrix equation for an axisymmetric triangular element.

Strain -Displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\text{Where, Co} - \text{ordinate}, r = \frac{r_1 + r_2 + r_3}{3}$$

$$z = \frac{z_1 + z_2 + z_3}{3}$$

18. Write down the Stress-Strain relationship matrix for an axisymmetric triangular element.

$$\text{Stress} - \text{Strain relationship matrix, [D]}$$

$$= \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

Where, E → Young's modulus

v → Poisson's ratio

19. Give the stiffness matrix equation for an axisymmetric triangular element.

Stiffness matrix [K] = 2π r A $[B]^T$ [D] [B]

$$\text{Where, Co} - \text{ordinate}, r \rightarrow \frac{r_1 + r_2 + r_3}{3}$$

A →Area of the triangular element matrix

20. What are the ways in which a three dimensional problem can be reduced to a two dimensional approach?

1. Plane stress: One dimension is too small when compared to other two dimensions.

Example: Gear - Thickness is small.

2. Plane strain: One dimension is too large when compared to other two dimensions.

   Example: Long pipe [Length is long compared to diameter]

3. Axisymmetric: Geometry is symmetric about the axis.

   Example: Cooling tower

UNIT 4

ISOPARAMETRIC ELEMENTS

4.1. ISOPARAMETRIC ELEMENTS

4.1.1. Introduction

The finite element method is a powerful technique for analysing engineering problems involving complex and irregular geometrics. However, the two and three dimensional elements (triangle, rectangle, brick) discussed in previous chapters cannot be used efficiently for irregular geometrics

Consider a continuum shown in Fig.4.1(a) and it is discretized by using triangular elements which is shown in Fig. 4.1(b)

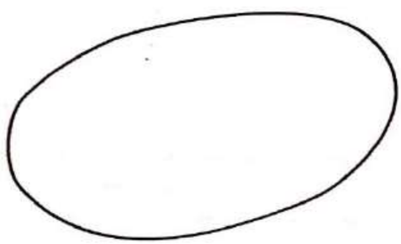


Fig.4.1(a) Continum

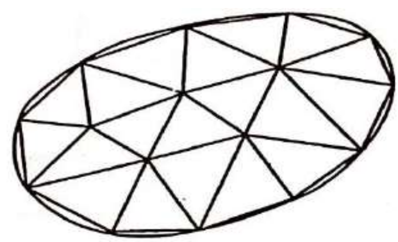


Fig.4.1(b) Continum is discretized by triangular elements

It is difficult to represent the curved boundaries by straight edges elements. A large number of elements may be used to obtain reasonable resemblance between original body and the assemblage. In order to overcome this drawback, isoparametric elements are used. I,e. for problems involving curved boundaries, a family of elements known as "isoparametric elements" can be used.

The isoparametric concept was first brought out by Taig and latter on generalized by B.M. Irons for mapping the curved boundaries. They brought out the concept of mapping for regular triangular, rectangular elements and brick elements from natural co-ordinate system to global cartesian system as shown in Fig.4.2, 4.3 & 4.4.

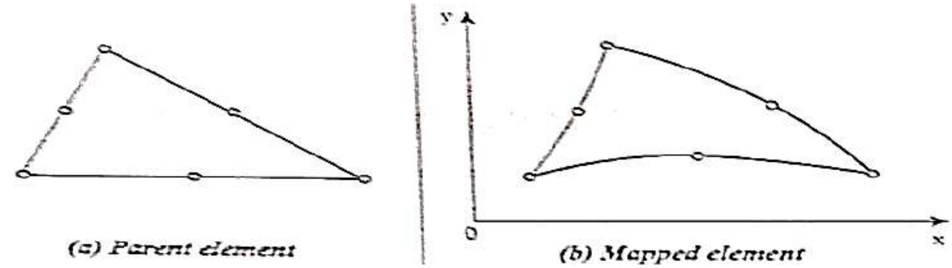


Fig.4.2. Concept of mapping in isoparametric elements (Triangular element)

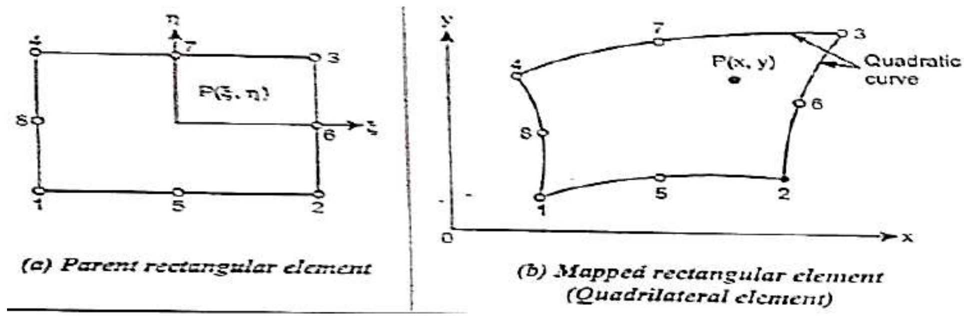


Fig.4.3

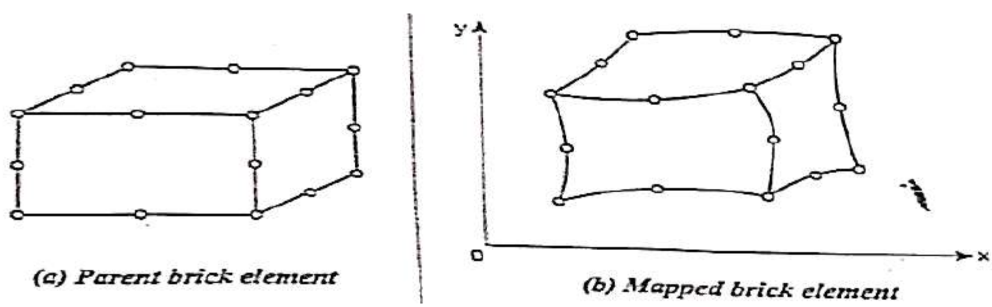


Fig.4.4

In this chapter, method of co-ordinate transformation from natural co-ordinate system to global co-ordinate system, shape function and stiffness matrix for four noded quadrilateral element, numerical integration are presented.

4.2. NATURAL CO-ORDINATES

A natural co-ordinate system is used to define any point inside the element by a set of dimensionless numbers whose magnitude never exceeds unity. This system is very useful in assembling of stiffness matrices.

(1) Natural Co-ordinates in One Dimension

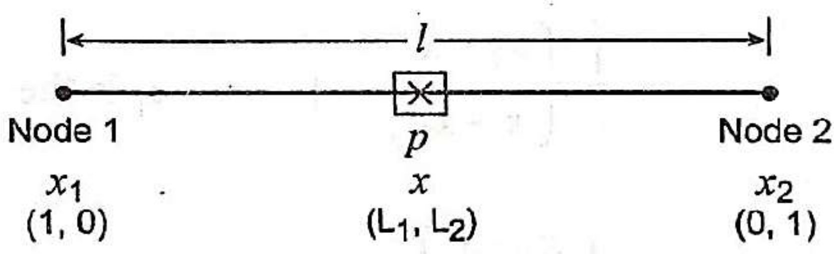


Fig.4.5 Natural Co-ordinates for a line element

Consider a two noded line element as shown in Fig.4.5. Any point p inside the line element is identified by two natural co-ordinates L, and L, and the cartesian co-ordinate x. Node 1 and node 2 have the cartesian co-ordinates x, and x, respectively.

We know that,

Total weightage of natural co-ordinates at any point is unity.

i.e, $$L_1 + L_2 = 1 \qquad ...(4.1)$$

Any point x within the element can be expressed as a linear combination of the nodal co-ordinates of nodes 1 and 2 as,

$$L_1 x_1 + L_2 x_2 = x$$

Arrange equation (5.1) and (5.2) in matrix form,

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$= \frac{1}{(x_2 - x_1\,)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\left(Note: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{(\,a_{11}.a_{22}) - (a_{12}.a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right)$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ -x_1 + x \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix}$$

$$= \frac{1}{l}\begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix} \therefore \text{x} - \text{x, is the length of the element, 1]}$$

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_2 - x}{l} \\ \dfrac{x - x_1}{l} \end{Bmatrix}$$

The variation of L, and L is shown in Fig.4.7 and Fig.4.8. L, is one at node 1 and it is zero at node 2 whereas L is one at node 2 and it is zero at node 1.

Fig.4.6

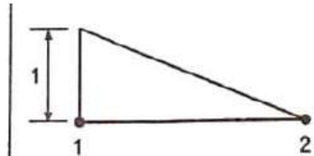


Fig.4.7 Variation of L₁

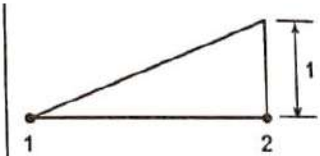


Fig.4.8 Variation of L₂

Integration of polynomial terms in natural co-ordinates can be performed by using the simple formula,

$$\int_{x_1}^{x_2} (L_1)^{\alpha}(L_2)^{\beta} dx = \frac{\alpha!\,\beta!}{(\alpha + \beta + 1)!} \times l_x$$

Where, α! is the factorial of α

Natural Co-ordinate, ε

In one dimensional problem, the following type of natural co-ordinate is also used. Consider a one dimensional element as shown in Fig.4.9.

In the local number scheme, the first node will be numbered 1 and the second node 2.e the centre of nodes 1 and 2 and p is the point referred.

The natural co-ordinator & for any point in the element is defined as,

$$\varepsilon = \frac{p\,c}{\left(\dfrac{x_2 - x_1}{2}\right)}$$

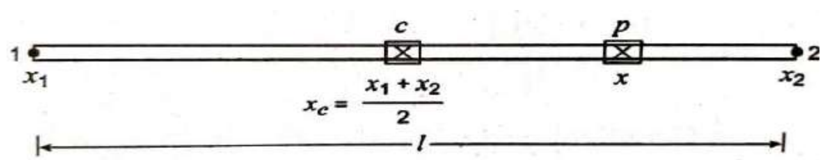


Fig.4.9.

$$\varepsilon = \frac{p\,c}{\frac{l}{2}} \; [\because x_2 - x_1 = l]$$

$$= \frac{2}{\mathrm{l}} p\, c = \frac{2}{\mathrm{l}}(\mathrm{x} - \mathrm{x_c}) \qquad [\because \mathrm{pc} = \mathrm{x} - \mathrm{xc}]$$

$$= \frac{2}{\mathrm{l}} \times \left[x - \left(\frac{x_1 + x_2}{2}\right)\right] \qquad \left[\because x_c = \frac{x_1 + x_2}{2}\right]$$

$$= \frac{2}{\mathrm{l}} \times \left[x - \left(\frac{x_2 + x_1}{2}\right)\right]$$

$$= \frac{2}{\mathrm{l}} \times \left[x - \left(\frac{x_2 - x_1 + 2x_1}{2}\right)\right]$$

$$= \frac{2}{\mathrm{l}} \times \left[x - \left(\frac{l + 2x_1}{2}\right)\right]$$

$$\varepsilon - \frac{2}{1} \times \left[x - \left(\frac{l}{2} + x_1\right)\right]$$

$$\frac{\varepsilon l}{2} = x - \frac{l}{2} - x_1$$

$$\frac{\varepsilon l}{2} + \frac{l}{2} = x - x_1$$

$$\boxed{\frac{l}{2}(\varepsilon + 1) = x - x_1}$$

Applying boundary conditions,

At node 1, $x = x_1$

$$\Rightarrow \qquad \frac{l}{2}(1 + \varepsilon) = 0$$

$$1 + \varepsilon = 0$$

$$\boxed{\varepsilon = -1}$$

At node 2, $x = x_2$

$$\Rightarrow \qquad \frac{l}{2}(1 + \varepsilon) = x_2 - x_1$$

$$\frac{l}{2}(1+\varepsilon) = 1$$

$$1+\varepsilon = 2$$

$$\boxed{\varepsilon = 1}$$

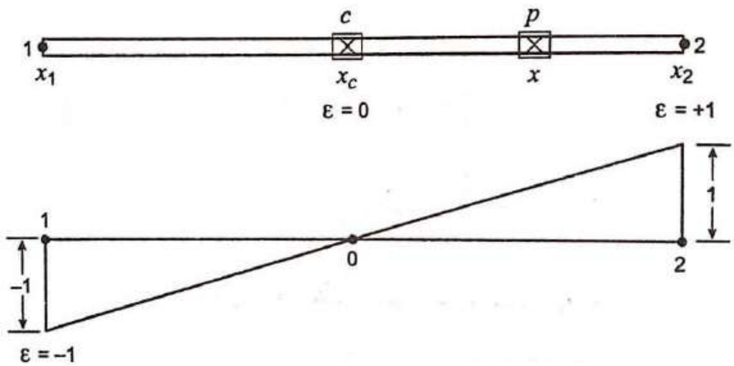


Fig.4.10. Variation of natural co-ordinates, ε

Natural Co-ordinates in Two Dimensions

Consider a triangular element having 3 nodes as shown in Fig.4.11.

Let p is the point inside the element and it has 3 co-ordinates L_1, L_2 and L_3.

From the definition of natural co-ordinates, we know that,

$$L_1 + L_2 + L_3 = 1$$

$$L_1x_1 + L_2x_2 + L_3x_3 = x$$
$$L_1y_1 + L_2y_2 + L_3y_3 = y$$

Assemble the above equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

Let $$D = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$D^{-1} = \frac{C^T}{|D|}$$

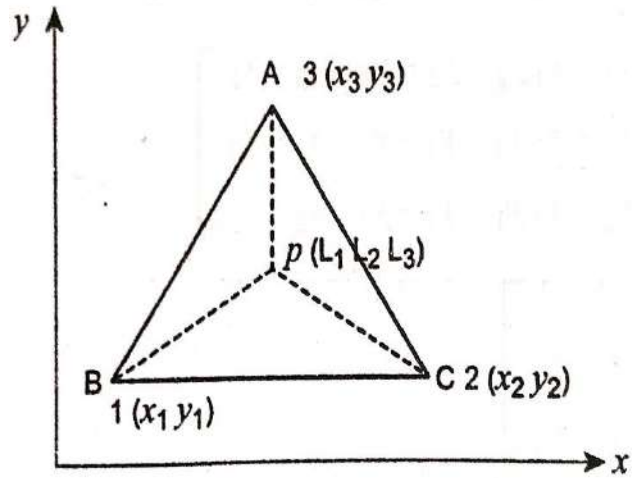


Fig.4.11

Coefficients of matrix D:

$$C_{11} = +\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = x_2y_3 - x_3y_2$$

$$C_{12} = -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} = -(x_1y_3 - x_3y_1) = x_3y_1 - x_1y_3$$

$$C_{13} = +\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ y_2 & y_3 \end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$C_{22} = +\begin{vmatrix} 1 & 1 \\ y_1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$C_{23} = -\begin{vmatrix} 1 & 1 \\ y_1 & y_2 \end{vmatrix} = -(y_2 - y_1) = y_1 - y_2$$

$$C_{31} = +\begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} = x_3 - x_2$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$C_{33} = + \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$C = \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$|D| = 1(x_2y_3 - x_3y_2) - 1(x_1y_3 - x_3y_1) + 1(x_1y_2 - x_2y_1)$$

Substitute C^T and $|D|$ values in equation

$$\Rightarrow \quad D^{-1} = \frac{1}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)} \times \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

Substitute $|D|$ values in equation,

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)} \times \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

The area of the triangle ABC can be expressed as a function of the x, y co-ordinates of the nodes 1, 2 and 3.

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\,[1(x_2y_3 - x_3y_2) - x_1\,(y_3 - y_2\,) + y_1\,(x_3 - x_2)]$$

$$= \frac{1}{2}[x_2y_3 - x_3y_2 - x_1\,y_3 + x_1y_2\, + x_3y_1\, - x_2y_1\,]$$

$$= \frac{1}{2}\,[x_2y_3 - x_3y_2 - (x_1\,y_3 - x_3y_1\,) + (x_1y_2\, - x_2y_1\,)]$$

$$(x_2y_3 - x_3y_2) - (x_1\,y_3 - x_3y_1\,) + (x_1y_2\, - x_2y_1\,) = 2\mathrm{A}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2\mathrm{A}} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

Integration of polynomial terms in natural co-ordinates for two dimensional elements can performed by using the formula,

$$\oint (L_1)^{\alpha} (L_2)^{\beta}\,(L_3)^{\gamma}\;dA = \frac{\alpha!\;\beta!\,\gamma!}{(\alpha + \beta + \gamma + 2)!} \times 2\mathrm{A}$$

4.3. SOLVED PROBLEMS ON NATURAL CO-ORDINATES

Example 4.1

Calculate the value of $\oint_A L_1L_2L_3\;dA$.

Solution:

$$\oint_{\mathrm{A}} \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3\;dA \qquad \text{... (1)}$$

We know that,

$$\oint (L_1)^{\alpha} (L_2)^{\beta}\,(L_3)^{\gamma}\;dA = \frac{\alpha!\;\beta!\,\gamma!}{(\alpha + \beta + \gamma + 2)!} \times 2\mathrm{A} \qquad \text{... (2)}$$

Compare equation (1) and (2),

$\alpha = 1, \beta = 1\; and\; \gamma = 1.$

$$\oint_{\mathrm{A}} \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3\;dA = \frac{1!\;\,1!\,1!}{(1 + 1 + 1 + 2)!} \times 2\mathrm{A}$$

$$= \frac{1 \times 1 \times 1}{5!} \times 2\mathrm{A}$$

$$= \frac{1}{5 \times 4 \times 3 \times 2 \times 1} \times 2\mathrm{A}$$

$$\oint_{\mathrm{A}} \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3 \, dA = \frac{\mathrm{A}}{60}$$

Example 4.2

Determine the value of $\oint_{A} \boldsymbol{L_1(L_2)^2(L_3)^{\,3}dA}$.

Solution: We know that,

$$\oint (L_1)^{\alpha}(L_2)^{\beta}\,(L_3)^{\gamma}\;dA = \frac{\alpha!\;\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!} \times 2\mathrm{A}$$

Here, $\alpha = 1$, $\beta = 2$ and $\gamma = 3$.

$$\oint_{\mathrm{A}} \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3 \, dA = \frac{1!\;\;2!\,3!}{(1+2+3+2)!} \times 2\mathrm{A} = \frac{1 \times 2 \times 3}{8!} \times 2\mathrm{A}$$

$$= \frac{1 \times 2 \times 1 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 2\mathrm{A}$$

$$\boxed{\oint_{\mathrm{A}} \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3 \, dA = \frac{\mathrm{A}}{1680}}$$

Example 4.3

Calculate the value of $\displaystyle\int_{\mathbf{0}}^{\mathbf{1}} \boldsymbol{L_1L_2dx}$.

Solution: We know that,

$$\int_{x_1}^{x_2} {L_1}^{\alpha}{L_2}^{\beta}dx = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!} \times\, l$$

Here $\alpha = 1, \beta = 1$.

$$\int_0^1 L_1 L_2 dx = \frac{1!\,1!}{(1+1+1)!} \times l = \frac{1}{3!} \times l$$

$$= \frac{1}{3 \times 2 \times 1} \times l$$

$$\boxed{\int_0^1 L_1 L_2 dx = \frac{1}{16}}$$

Example 4.4

Determine the value of $\int_0^1 L_1{}^3\, dx$

Solution: We know that,

$$\int_{x_1}^{x_2} L_1{}^{\alpha} L_2{}^{\beta}\, dx = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!} \times l$$

Here, $\alpha = 3, \beta = 0$

$$\int_0^1 L_1{}^3\, L_2{}^0 dx = \frac{3!\,0!}{(3+0+1)!} \times l = \frac{(3 \times 2 \times 1)}{4!} \times l$$

$$= \frac{3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \times l$$

$$\boxed{\int_0^1 L_1{}^3\, L_2{}^0 dx = \frac{\mathrm{l}}{4}}$$

4.4. ISOPARAMETRIC, SUPERPARAMETRIC AND SUBPARAMETRIC ELEMENTS

Isoparametric Element

We know that, shape functions are used for defining the geometry and displacements of the element. Consider a element shown in Fig.4.12.

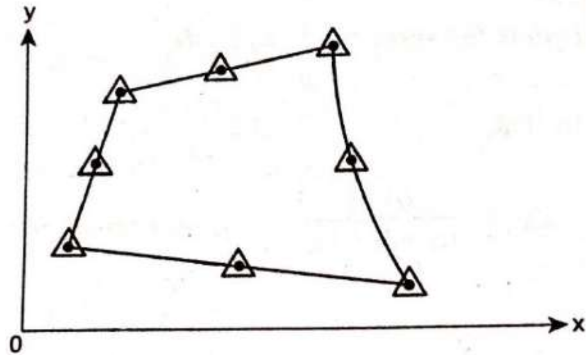


Fig.4.12.

• Nodes used for defining geometry.

A Nodes used for defining displacements.

In this element, all the eight nodes are used in defining geometry as well as displacements.

If the number of nodes used for defining the geometry is same as number of for defining the displacements, then, it is known as isoparametric element.

Superparametric Element

Consider a element shown in Fig. 4.13.

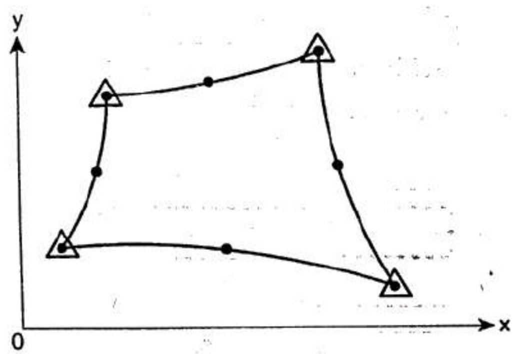


Fig.4.13.

- Nodes used for defining geometry.
- A Nodes used for defining displacements.

In this element, eight nodes are used to define the geometry and four nodes are used to define the displacements. If the number of nodes used for defining the geometry is more than number of nodes used for defining the displacements, then, it is known as superparametric element.

Subparametric Element

Consider a element shown in Fig.4.14.

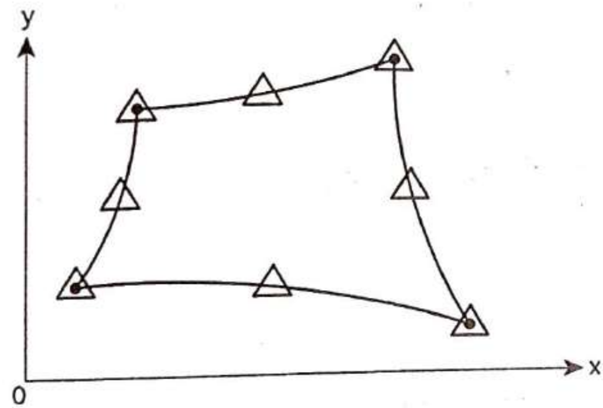


Fig.4.14

- • Nodes used for defining geometry.
- A Nodes used for defining displacements.

In this element, four nodes are used to define the geometry and eight nodes are used to define the displacements. If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacements, then it is known as subparametric element.

4.5. ONE DIMENSIONAL SHAPE FUNCTIONS FOR ISOPARAMETRIC FORMULATION OF THE BAR ELEMENT

Consider a bar element with nodes 1 and 2 as shown in Fig.4.15. u_1 and u_2 are the displacements at the respective nodes. So, u_1 and u_2 are considered as degree of freedom of this bar element.

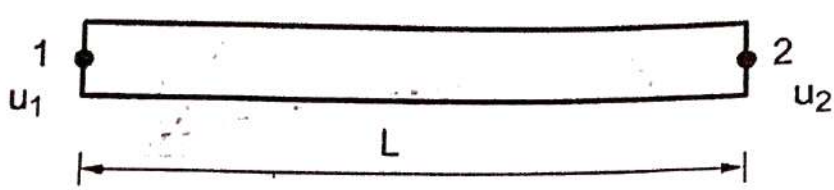


Fig. 4.15 Linear bar element

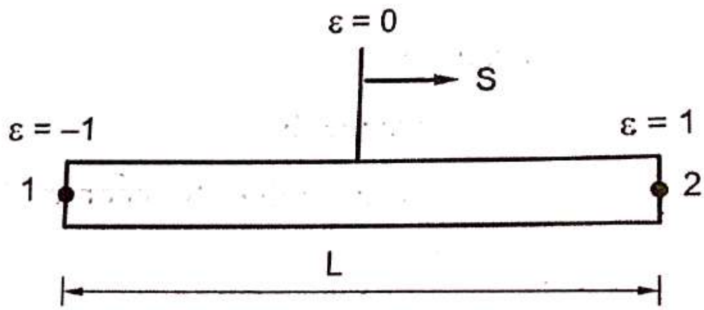


Fig.4.16 Natural co-ordinate system

First, the natural coordinate "ε" is attached to the element, with the origin located at the centre of the element, as shown in Fig.4.16.

The "ε" axis need not be parallel to the x-axis - this is only for convenience.

We consider the bar element to have two degrees of freedom-axial displacements u_1 and u_2 at each node associated with the global axis.

When the "ε" and "x" axes are parallel to each other, the "ε" and "x" coordinates can be related by,

$$x = x_e + \frac{L}{2}\varepsilon \qquad \text{... (4.4)}$$

Where $x_e \rightarrow$ is the global coordinate of the element centroid using the global coordinates x_1 *and* x_2.

$$x_e = \frac{(x_1 + x_2)}{2} \qquad \text{... (4.5)}$$

Substitute the equation (4.5) in equation (4.4),

$$x = \frac{x_1 + x_2}{2} + \frac{L}{2}\varepsilon$$

we know that $x = \dfrac{x_1 + x_2}{2} + \dfrac{L}{2}\varepsilon \qquad [\because L = (x_2 - x_1)]$

$$x = \left(\frac{x_1 + x_2}{2}\right) + \left(\frac{x_2 + x_1}{2}\right)\varepsilon$$

$$\left(\frac{x_2 + x_1}{2}\right)\varepsilon = x - \left(\frac{x_1 + x_2}{2}\right)$$

We can express the natural coordinate 'ε' in terms of the global coordinates as,

$$\varepsilon = \left[x - \frac{(x_1 + x_2)}{2}\right] \times \frac{2}{(x_2 - x_1)} \qquad \text{... (4.6)}$$

Since, the element has got two degrees of freedom, it will have two generalized coordinates,

$$x = a_1 + a_2\varepsilon \qquad \text{.... (4.7)}$$

Where, ε is such that -1 ≤ ε ≤ 1. Solving for the $a_i's$ in terms of x_1 *and* x_2, we obtain

$$x = \frac{1}{2}[(1 - \varepsilon)x_1 + (1 + \varepsilon)x_2] \qquad \text{.... (4.8)}$$

Writing the equation (5.19) in matrix form,

$$\{x\} = \frac{1}{2}[(1 - \ \varepsilon)(1 + \ \varepsilon)]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

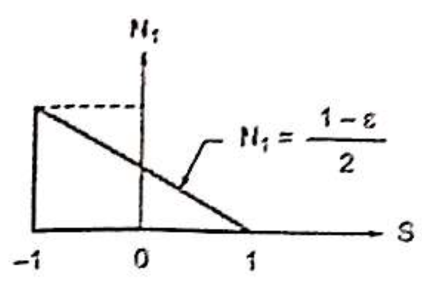
$$\{x\} = [N_1 \ N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Displacement function, $x = N_1 \, u_1 + N_2 \, u_2$

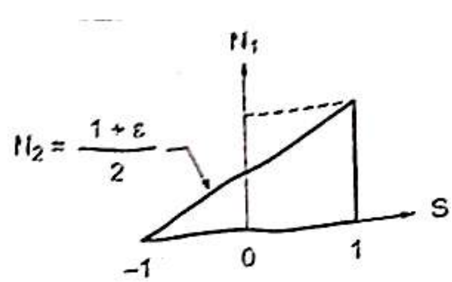
where, shape function, $\quad N_1 = \dfrac{1 - \varepsilon}{2}$

$$N_2 = \frac{1 + \varepsilon}{2}$$

We may note, that N, and N_2 obey the definition of shape function i.e., the shape function will have a value equal to unity at the node to which it belongs and zero value at other



(a)



(b)

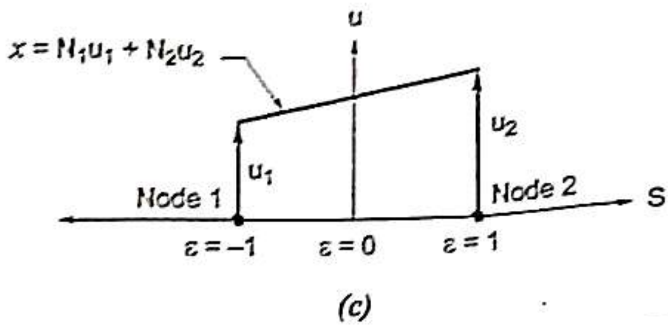


Fig. 4.17. Shape function variations with natural coordinates (a) shape function N_1, (b) shape function N_2, and (c) Linear displacement field

4.6. SHAPE FUNCTIONS FOR 4 NODED RECTANGULAR PARENT ELEMENT BY USING NATURAL CO-ORDINATE SYSTEM AND CO-ORDINATE TRANSFORMATION [TWO DIMENSIONAL]

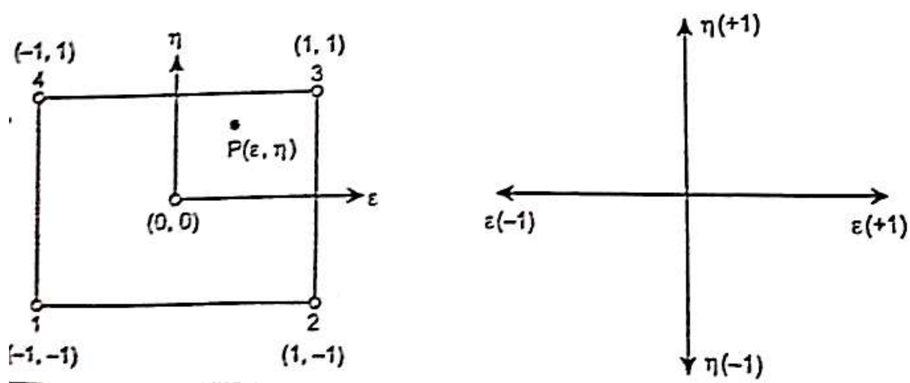


Fig. 4.18. Four noded rectangular parent element

Consider a four noded rectangular element as shown in Fig 4.18. The parent element defined in e and co-ordinates Le, natural co-ordinates e is varying from -1 to 1 and is also varying-1 to 1.

We know that,

Shape function value is unity at its node and its value is zero at other nodes.

At node 1: (Co-ordinates $\varepsilon = -1$, $\eta = -1$)

Shape function, $N_1 = 1$ at node 1.

$N_1 = 0$ at nodes 2, 3, and 4.

N_1 has to be in the form of $N_1 = C\,(1 - \varepsilon)\,(1 - \eta)$ …(4.20)

where, C is constant.

Substitute $\varepsilon = -1$ and $\eta = -1$ in equation (4.20).

$\Rightarrow N_1 = C\,(1+1)(1+1)$

$\Rightarrow N_1 = 4C$

$\Rightarrow 1 = 4C$

$$\Rightarrow C = \frac{1}{4}$$

Substitute C value in equation (4.20),

$$N_1 = \frac{1}{4}(1 - \varepsilon)\,(1 - \eta) \qquad \text{... (4.21)}$$

At node 2: (Co-ordinates $\varepsilon = 1$, $\eta = -1$)

Shape function $N_2 = 1$ at node 2.

$N_1 = 0$ at node 1, 3 and 4.

N_2 has to be in form of, $N_2 = C\,(1+\varepsilon)\,(1-\eta)$ …(4.22)

Substitute $\varepsilon = 1$ and $\eta = -1$ in equation (4.22),

$N_2 = C(1+1)\,(1+1)$

$N_2 = 4C$

$1 = 4C \qquad [\because N_2 = 1]$

$$C = \frac{1}{4}$$

Substitute C value in equation (4.22)

$$\Rightarrow \qquad N_2 = \frac{1}{4}(1 + \varepsilon)(1 - \eta) \qquad \text{... (4.23)}$$

At node 3: (Co – ordinates $\varepsilon = 1$, $\eta = 1$)

Shape functions $N_3 = 1$ at node 3.

$N_3 = 0$ at node 1, 2 and 4.

N_3 has to be in the from of $N_3 = C(1+\varepsilon)(1+\eta)$ …(4.24)

Substitute $\varepsilon = 1$ and $\eta = 1$ in equation (4.24)

$$\Rightarrow \quad N_3 = C(1+1)(1+1)$$

$$N_3 = 4\,C$$

$$1 = 4\,C \qquad [\because N_3 = 1]$$

$$\Rightarrow \quad C = \frac{1}{4}$$

Substitute C value in equation (4.24)

$$\Rightarrow \quad N_3 = \frac{1}{4}(1+\varepsilon)(1-\eta) \qquad \text{... (4.25)}$$

At node 4: (Co – ordinates $\varepsilon = -1$, $\eta = 1$)

Shape functions $N_4 = 1$ at node 4.

$N_4 = 0$ at node 1, 2 and 3

N_4 has to be in the from of $N_4 = C(1-\varepsilon)(1+\eta)$ …(4.26)

Substitute $\varepsilon = -1$ and $\eta = 1$ in equation (4.26)

$$\Rightarrow \quad N_4 = C(1+1)(1+1)$$

$$N_4 = 4\,C$$

$$1 = 4\,C \qquad [\because N_4 = 1]$$

$$\Rightarrow C = \frac{1}{4}$$

Substitute C value in equation (4.26)

$$\Rightarrow \quad N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta) \qquad \text{... (4.27)}$$

Consider a point p with co-ordinate (ε, η). If displacement function $u = \begin{Bmatrix} u \\ v \end{Bmatrix}$ represents the displacement components of a point located at (ε, η) then,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

and

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

It can be written in matrix form as,

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \ldots (4.28)$$

In the isoparametric formulation i.e., for global system, the co-ordinates of the nodal points are $(x_1, y_1), (x_2, y_2), (x_3, y_3) and\ (x_4, y_4)$. In order to get mapping, the co-ordinate of point *p* is defined as

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

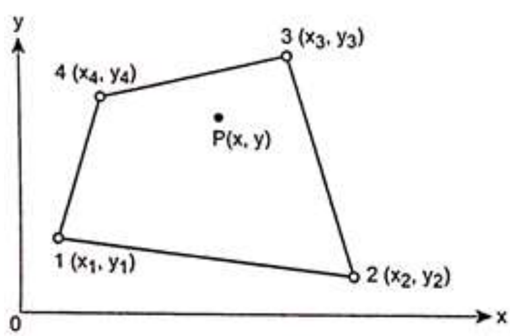


Fig 4.19. Four noded quadrilateral element or mapped element

The above equations can be written in matrix form as,

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \quad \ldots (4.29)$$

4.7 ELEMENT STIFFNESS MATRIX EQUATION FOR 4 NODED ISOPARAMETRIC QUADRILATERAL ELEMENT

Assembling element stiffness matrix for isoparametric element is a tedious process since it involves co-ordinate transformation from natural co-ordinate system to global co-ordinate system.

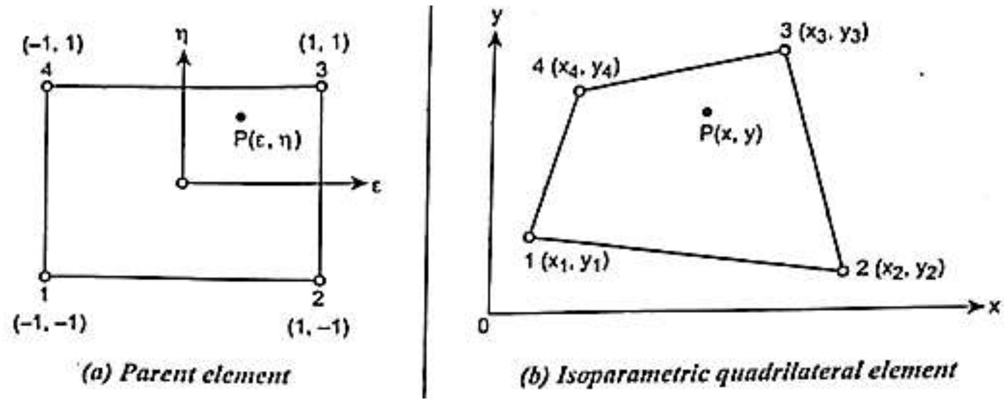


Fig. 4.20

The displacement function u for parent rectangular element is given by,

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

The displacement function u for parent rectangular element is given by,

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

We have to express the derivatives of a function in x, y co-ordinates in terms of its derivatives in ε, η co-ordinates. This can be done as follows:

$$Let \qquad f = f(x, y)$$

$$f = f[x(\varepsilon, \eta), y(\varepsilon, \eta)]$$

The relationship between natural co-ordinates and global co-ordinates can be calculated by using chain rule of partial differentiation.

$$\Rightarrow \qquad \frac{\partial f}{\partial \varepsilon} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \varepsilon}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \eta}$$

Arranging the above equations in matrix from,

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \varepsilon} & \dfrac{\partial y}{\partial \varepsilon} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} \qquad \text{... (4.30)}$$

Where J is the Jacobain matrix

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \varepsilon} & \dfrac{\partial y}{\partial \varepsilon} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \qquad \text{... (4.31)}$$

Where, $J_{11} \dfrac{\partial x}{\partial \varepsilon}; J_{12} = \dfrac{\partial y}{\partial \varepsilon}$

$$J_{21} \frac{\partial x}{\partial \eta}; J_{22} = \frac{\partial y}{\partial \eta}$$

We know that,

$$\left.\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \end{aligned}\right\} \qquad \text{... (4.32)}$$

$$J_{11} = \frac{\partial x}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon} x_1 + \frac{\partial N_2}{\partial \varepsilon} x_2 + \frac{\partial N_3}{\partial \varepsilon} x_3 + \frac{\partial N_4}{\partial \varepsilon} x_4 \qquad \text{... (4.33)}$$

$$J_{12} = \frac{\partial y}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon} y_1 + \frac{\partial N_2}{\partial \varepsilon} y_2 + \frac{\partial N_3}{\partial \varepsilon} y_3 + \frac{\partial N_4}{\partial \varepsilon} y_4 \qquad \text{... (4.34)}$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4 \qquad \text{... (4.35)}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 \qquad \text{... (4.36)}$$

We know that,

Shape functions for quadrilateral element are:

$$N_1 = \frac{1}{4}(1 - \varepsilon)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 + \varepsilon)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \varepsilon)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \varepsilon)(1 + \eta)$$

[From equations (4.21), (4.23), (4.25) and (4.27)]

$$\Rightarrow \qquad \frac{\partial N_1}{\partial \varepsilon} = \frac{1}{4}(-1) \times (1 - \eta) = \frac{1}{4} \times -(1 - \eta) \qquad \text{... (4.37)}$$

$$\Rightarrow \qquad \frac{\partial N_2}{\partial \varepsilon} = \frac{1}{4}(1) \times (1 - \eta) = \frac{1}{4} \times (1 - \eta) \qquad \text{... (4.38)}$$

$$\Rightarrow \qquad \frac{\partial N_3}{\partial \varepsilon} = \frac{1}{4}(1) \times (1 + \eta) = \frac{1}{4}(1 + \eta) \qquad \text{... (4.39)}$$

$$\Rightarrow \qquad \frac{\partial N_4}{\partial \varepsilon} = \frac{1}{4}(-1) \times (1 + \eta) = \frac{1}{4} \times -(1 + \eta) \qquad \text{... (4.40)}$$

$$\Rightarrow \qquad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1 - \varepsilon)(-1) = \frac{1}{4} \times -(1 - \varepsilon) \qquad \text{... (4.41)}$$

$$\Rightarrow \qquad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\varepsilon)(-1) = \frac{1}{4} \times -(1+\varepsilon) \qquad \text{... (4.42)}$$

$$\Rightarrow \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\varepsilon)(1) = \frac{1}{4} \times (1+\ \varepsilon) \qquad \text{... (4.43)}$$

$$\Rightarrow \qquad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\varepsilon)(1) = \frac{1}{4} \times (1+\ \varepsilon) \qquad \text{... (4.44)}$$

Substitute $\frac{\partial \mathrm{N}_1}{\partial \varepsilon}, \frac{\partial \mathrm{N}_2}{\partial \varepsilon}, \frac{\partial \mathrm{N}_3}{\partial \varepsilon}, \frac{\partial \mathrm{N}_4}{\partial \varepsilon}, \frac{\partial \mathrm{N}_1}{\partial \eta}, \frac{\partial \mathrm{N}_2}{\partial \eta}, \frac{\partial \mathrm{N}_3}{\partial \eta}$, and $\frac{\partial \mathrm{N}_4}{\partial \eta}$ values in equaitons

Equation (4.33) becomes,

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\ \eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \quad \text{... (4.45)}$$

Equation (4.34) becomes,

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\ \eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \quad \text{... (4.46)}$$

Equation (4.35) becomes,

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4] \quad \text{... (4.47)}$$

Equation (4.36) becomes,

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4] \quad \text{... (4.48)}$$

From equation (4.31), we know that,

$$\text{Jacobain matrix, } [\,J\,] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \qquad \text{... (4.49)}$$

Where,

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\ \eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\ \eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4]$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4]$$

From equation (4.30), we know that,

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \varepsilon} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \qquad \ldots (4.50)$$

The strain-displacement relations are,

$$e = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad \ldots (4.51)$$

Substituting $f = u$ in equation (5.50).

$$\Rightarrow \qquad \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \varepsilon} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \qquad \ldots (4.52)$$

$$Similarly \Rightarrow \begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \varepsilon} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \qquad \text{... (4.53)}$$

Equations (4.51), (4.52) and (4.53) yield,

$$\text{Strain}, \{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \varepsilon} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \varepsilon} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \qquad \text{... (4.54)}$$

We know that, $u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$

$y = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$

$$\Rightarrow \quad \frac{\partial u}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon}u_1 + \frac{\partial N_2}{\partial \varepsilon}u_2 + \frac{\partial N_3}{\partial \varepsilon}u_3 + \frac{\partial N_4}{\partial \varepsilon}u_4 \qquad \text{... (4.55)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial \eta} = \frac{\partial N_1}{\partial \eta}u_1 + \frac{\partial N_2}{\partial \eta}u_2 + \frac{\partial N_3}{\partial \eta}u_3 + \frac{\partial N_4}{\partial \eta}u_4 \qquad \text{... (4.56)}$$

$$\Rightarrow \quad \frac{\partial v}{\partial \varepsilon} = \frac{\partial N_1}{\partial \varepsilon}v_1 + \frac{\partial N_2}{\partial \varepsilon}v_2 + \frac{\partial N_3}{\partial \varepsilon}v_3 + \frac{\partial N_4}{\partial \varepsilon}v_4 \qquad \text{... (4.57)}$$

$$\Rightarrow \quad \frac{\partial v}{\partial \eta} = \frac{\partial N_1}{\partial \eta}v_1 + \frac{\partial N_2}{\partial \eta}v_2 + \frac{\partial N_3}{\partial \eta}v_3 + \frac{\partial N_4}{\partial \eta}v_4 \qquad \text{... (4.58)}$$

Assembling the equations (4.55), (4.56), (4.57) & (4.58) in matrix form,

$$\Rightarrow \quad \begin{Bmatrix} \dfrac{\partial u}{\partial \varepsilon} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \varepsilon} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \begin{bmatrix} \dfrac{\partial N_1}{\partial \varepsilon} & 0 & \dfrac{\partial N_2}{\partial \varepsilon} & 0 & \dfrac{\partial N_3}{\partial \varepsilon} & 0 & \dfrac{\partial N_4}{\partial \varepsilon} & 0 \\ \dfrac{\partial N_1}{\partial \eta} & 0 & \dfrac{\partial N_2}{\partial \eta} & 0 & \dfrac{\partial N_3}{\partial \eta} & 0 & \dfrac{\partial N_4}{\partial \eta} & 0 \\ 0 & \dfrac{\partial N_1}{\partial \varepsilon} & 0 & \dfrac{\partial N_2}{\partial \varepsilon} & 0 & \dfrac{\partial N_3}{\partial \varepsilon} & 0 & \dfrac{\partial N_4}{\partial \varepsilon} \\ 0 & \dfrac{\partial N_1}{\partial \eta} & 0 & \dfrac{\partial N_2}{\partial \eta} & 0 & \dfrac{\partial N_3}{\partial \eta} & 0 & \dfrac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{... (4.59)}$$

$$\Rightarrow Strain, \{e\} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \text{... (4.60)}$$

We know that, $Strain, \{e\} = [\,B\,]\ \{\,u\,\}$

$$\Rightarrow \qquad \{e\} = [\,B\,] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \text{... (4.61)}$$

Comparing equations (4.60) and (4.61),

Strain − Displacement matrix$[\,B\,]$

$$= \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \text{... (4.62)}$$

Substitute $\frac{\partial N_1}{\partial \varepsilon}, \frac{\partial N_2}{\partial \varepsilon}, \frac{\partial N_3}{\partial \varepsilon}, \frac{\partial N_4}{\partial \varepsilon}, \frac{\partial N_1}{\partial \eta}, \frac{\partial N_2}{\partial \eta}, \frac{\partial N_3}{\partial \eta}$, and $\frac{\partial N_4}{\partial \eta}$ values in equation (4.62),

[Refer equations (4.37) to (4.44)]

$$\Rightarrow [\,B\,] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\times \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix} \quad \ldots (4.63)$$

This is a strain-displacement relationship matrix [B] equation for isoparametric quadrilateral element.

We know that, General element stiffness matrix equation,

$$[\,K\,] = \oint [\,B]^T [D][B]^{dv}$$

For Isoparametric quadrilateral element,

$$\Rightarrow \qquad [\,K\,] = t \iint [\,B]^T [D]\,[B]\ \partial x\,\partial y \qquad \ldots (4.64)$$

$$[\because \partial x\,\partial y = |\,J|\ \partial \varepsilon\,\partial \eta]$$

Where, t → Thickness of the element

| J | → Determinant of the J

ε, η → Natural co-ordinates

[B] → Strain- Displacement relationship matrix

[D] → Stress- Strain relationship matrix

For two dimensional problems,

Stress-Strain relationship matrix,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \text{[For plane stress conditions]}$$

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

[For plane stress conditions]

Where, E → Young's modulus

v → Poisson's ratio

4.8 ELEMENT FORCE VECTOR

The element force vector is given by,

$$\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \quad \text{... (4.65)}$$

Where, N is the shape function.

F_x is a load or force on x direction.

F_y is a force on y direction.

4.9 SOLVED PROBLEMS – ISOPARAMETRIC ELEMENTS

Example.4.5

Evaluate the Cartesian co-ordinate of the point P which has local co-ordinates ε = 0.6 and η = 0.8 as shown in Fig. (i).

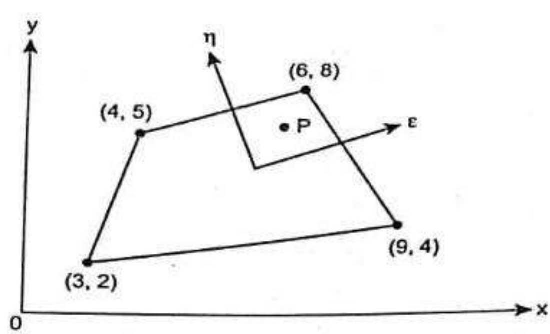


Fig. (i).

Given: Natural co-ordinates of point P

ε = 0.6

$\eta = 0.8$

Cartesian co-ordinates of point 1, 2, 3 and 4.

$x_1 = 3; \qquad y_1 = 2$

$x_2 = 9; \qquad y_2 = 4$

$x_3 = 6; \qquad y_3 = 8$

$x_4 = 4; \qquad y_4 = 5$

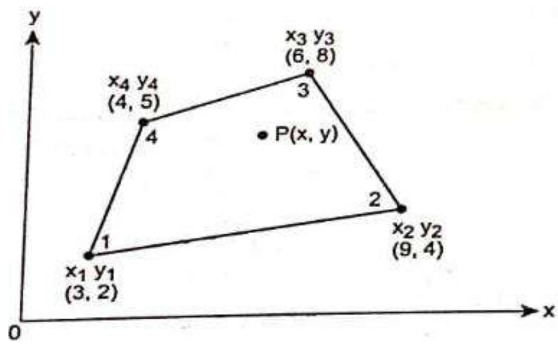


Fig. (ii).

To find: Cartesian co-ordinates of the point P(x, y).

Solution: We know that,

Shape functions for quadrilateral element are:

$$N_1 = \frac{1}{4}(1 - \varepsilon)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 + \varepsilon)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \varepsilon)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \varepsilon)(1 + \eta)$$

Substitute ε and η values in the above equations,

$$\Rightarrow \qquad N_1 = \frac{1}{4}(1 - 0.6)(1 - 0.8) = 0.02$$

$$\Rightarrow \qquad N_2 = \frac{1}{4}(1 + 0.6)(1 - 0.8) = 0.08$$

$$\Rightarrow \qquad N_3 = \frac{1}{4}(1 + 0.6)(1 + 0.8) = 0.72$$

$$\Rightarrow \qquad N_4 = \frac{1}{4}(1 - 0.6)(1 + 0.8) = 0.18$$

We know that,

co − ordinate, $x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$

$= 0.02\ (3) + 0.08\ (9) + 0.72\ (6) + 0.18(4)$

$x = 5.82$

Similarly, Co-ordinate, $y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$

$= 0.02 \times (2) + 0.08\ (4) + 0.72\ (8) + 0.18(5)$

$y = 7.02$

Result: The Cartesian co-ordinates of point P are $\overset{x}{(5.82}, \overset{y}{7.02)}$.

Example 4.6

For the isoparametric four noded quadrilateral element shown in fig.(i), determine the Cartesian co-ordinates of point P which has local co-ordinates $\varepsilon = 0.5$ and $\eta = 0.5$.

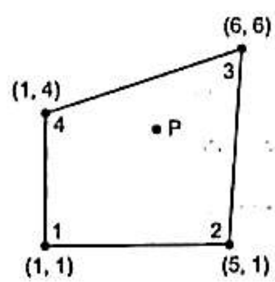


Fig. (i)

Given : Natural co-ordinates of point P

$\varepsilon = 0.5$

$\eta = 0.5$

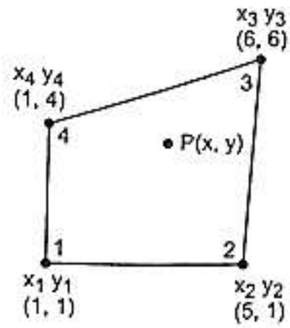


Fig (ii)

Cartesian co-ordinates of point 1, 2, 3 and 4,

$$x_1 = 1; \qquad y_1 = 1$$

$$x_2 = 5; \qquad y_2 = 1$$

$$x_3 = 6; \qquad y_3 = 6$$

$$x_4 = 1; \qquad y_4 = 4$$

To find: Cartesian co-ordinates of point P (*x, y*).

Solution: we know that,

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1 - \varepsilon)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 + \varepsilon)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \varepsilon)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \varepsilon)(1 + \eta)$$

Substitute ε and η values in the above equations,

$$\Rightarrow \qquad N_1 = \frac{1}{4}(1 - 0.5)(1 - 0.5) = 0.0625$$

$$\Rightarrow \qquad N_2 = \frac{1}{4}(1 + 0.5)(1 - 0.5) = 0.1875$$

$$\Rightarrow \qquad N_3 = \frac{1}{4}(1 + 0.5)(1 + 0.5) = 0.5625$$

$$\Rightarrow \qquad N_4 = \frac{1}{4}(1 - 0.5)(1 + 0.5) = 0.1875$$

We know that,

$$\text{co} - \text{ordinate}, x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$= 0.0625 \times 1 + 0.1875 \times 5 + 0.5625 \times 6 + 0.1875 \times 1$$

$$x = 4.5625$$

Similarly,

$$\text{Co-ordinate,} \quad y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

$$= 0.0625 \times 1 + 0.1875 \times 1 + 0.5625 \times 6 + 0.1875 \times 4$$

$$y = 4.375$$

x y

Result: The Cartesian co-ordinates of point P are (4.5625, 4.375).

Example 4.7

For the isoparametric quadrilateral element shown in Fig.(i), determine the local co-ordinates of the point P which has cartestian co-ordinates (7,4).

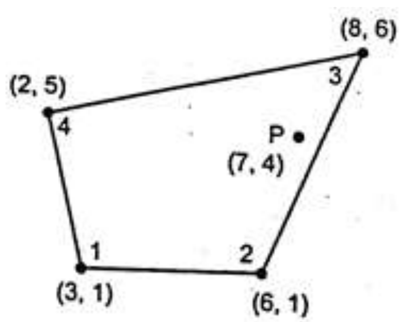


Fig. (i)

Given:

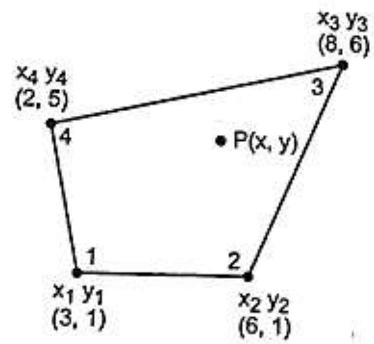


Fig. (ii)

Cartesian co-ordinates of point P,

$x = 7; \qquad y = 4$

Cartesian co-ordinates of point 1, 2, 3 and 4,

$$x_1 = 1; \qquad y_1 = 1$$

$$x_2 = 5; \qquad y_2 = 1$$

$$x_3 = 6; \qquad y_3 = 6$$

$$x_4 = 1; \qquad y_4 = 4$$

To find: Local co-ordinates of the point P, i.e., ε and η.

Solution: we know that,

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1 - \varepsilon)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 + \varepsilon)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \varepsilon)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \varepsilon)(1 + \eta)$$

Cartesian co − ordinate of point $P(x, y)$,

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \quad \dots(1)$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \quad \dots(2)$$

Substitute $N_1, N_2, N_3, N_4, x_1, x_2, x_3$ *and* x_4 values in equations (1),

$$\Rightarrow \quad 7 = \frac{1}{4}[(1- \varepsilon)(1-\eta) \times 3 + (1+\varepsilon)(1-\eta) \times 6(1+\varepsilon)(1+\eta) \times 8 + (1 - \varepsilon)(1+\eta) \times 2]$$

$$\Rightarrow \quad 28 = [(1-\eta- \varepsilon+ \varepsilon\eta)3 + (1-\eta+\varepsilon-\varepsilon\eta)6 + (1+\eta+\varepsilon+\varepsilon\eta)8\,(1+\eta- \varepsilon-\varepsilon\eta)2]$$

$$\Rightarrow \quad 28 = 3 - 3\eta - 3\varepsilon + 3\varepsilon\eta + 6 - 6\eta + 6\varepsilon - 6\varepsilon\eta + 8 + 8\eta + 8\varepsilon + 8\varepsilon\eta + 2 + 2\eta - 2\varepsilon - 2\varepsilon\eta$$

$$28 = 19 + \eta + 9\varepsilon + 3\varepsilon\eta$$

$$\Rightarrow \quad \eta + 9\varepsilon + 3\varepsilon\eta = 9 \quad \dots.(3)$$

subsitute $\mathrm{N_1, N_2, N_3, N_4, y, y_1, y_2, y_3}$ and $\mathrm{y_4}$vlaues in equaiton (2),

$$4 = \frac{1}{4}[(1- \varepsilon)(1-\eta) \times 1 + (1+\varepsilon)(1-\eta) \times 1 + (1+\varepsilon)(1+\eta) \times 6 + (1-\varepsilon)(1+\eta) \times 5]$$

$$= \frac{1}{4}[1- \eta - \varepsilon + \varepsilon\eta + 1 - \eta + \varepsilon - \varepsilon\eta + 6 + 6\eta + 6\varepsilon + 6\varepsilon\eta + 5 + 5\eta - 5\varepsilon - 5\varepsilon\eta]$$

$$4 = \frac{1}{4}[13 + 9\eta + \varepsilon + \varepsilon\eta]$$

$$\Rightarrow \quad 16 = 13 + 9\eta + \varepsilon + \varepsilon\eta$$

$$\Rightarrow \quad 9\eta + \varepsilon + \varepsilon\eta = 3 \quad \dots(4)$$

Equation (4) multiplied by (-3),

$$-27\,\eta - 3\,\varepsilon - 3\varepsilon\eta = -9 \qquad \dots (5)$$

Solving equation (3) and (5)

$$\eta + 9\varepsilon + 3\varepsilon\eta = 9$$

$$-27\,\eta - 3\,\varepsilon - 3\varepsilon\eta = -9$$

$$-26\,\eta + 6\,\varepsilon = 0$$

$$-26\,\eta = -6\,\varepsilon$$

$$\Rightarrow \qquad \varepsilon = 4.3333\,\eta \qquad \dots (6)$$

Substitute ε value in equation (3),

$$(3) \Rightarrow \eta + 9\,(4.3333\,\eta) + 3\,(4.3333\,\eta) \times \eta = 9$$

$$\eta + 39\eta + 13\,\eta^2 = 9$$

$$\Rightarrow \qquad 13\,\eta^2 + 40\eta = 9$$

$$\Rightarrow \qquad 13\,\eta^2 + 40\eta - 9 = 0$$

$$\eta = \frac{-40 \pm \sqrt{(40)^2 - 4(13)(-9)}}{2(13)}\eta$$

$$\left[a\,x^2 + b\,x + c = 0; Roots : \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}\right]$$

$$= \frac{-40 + 45.475}{26}$$

$$\eta = 0.210587$$

Substitute η value in equation (6),

$$\Rightarrow \qquad \varepsilon = 4.33333 \times 0.210587$$

$$\varepsilon = 0.912545$$

Result : Local co-ordinates of the point P,

$$\eta = 0.210587$$

$$\varepsilon = 0.912545$$

Example 4.8

Evaluate [J] at ε = η = ½ for the linear quadrilateral element shown in fig.(i).

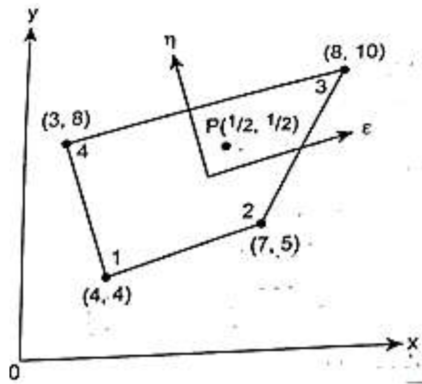


Fig. (i)

Given: Natural co-ordinates at point, P.

$$\varepsilon = \frac{1}{2} = 0.5\,; \qquad \eta = \frac{1}{2} = 0.5$$

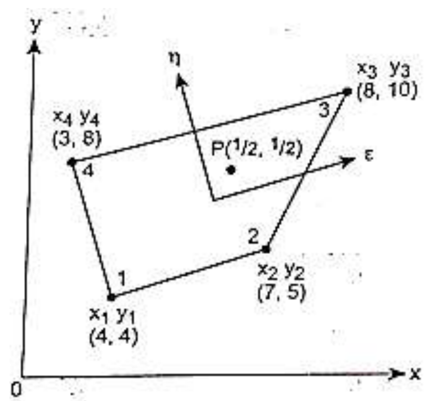


Fig. (ii)

Cartesian co-ordinates of point 1, 2, 3 and 4,

$$x_1 = 4; \qquad y_1 = 4$$

$$x_2 = 7; \qquad y_2 = 5$$

$$x_3 = 8; \qquad y_3 = 10$$

$$x_4 = 3; \qquad y_4 = 8$$

To find: 1 Jacobian matrix [J]

Solution: Jacobian matrix for quadrillaternal element is given by,

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \quad ...(1)$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \quad ...(2)$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4] \quad ...(3)$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4] \quad ...(4)$$

Subsitutute $\eta, \varepsilon, x_1, x_2, x_3, x_4, y_1, y_2, y_3$ and y_4 values in equaiton (1), (2), (3)and (4).

(1)⇒ $$J_{11} = \frac{1}{4}[-(1-0.5)4 + (1-0.5)7 + (1+0.5)8 - (1+0.5)3]$$

$$J_{11} = 2.25$$

(2)⇒ $$J_{12} = \frac{1}{4}[-(1-0.5)\times 4 + (1-0.5)\times 5 + (1+0.5)10 - (1+0.5)8]$$

$$J_{12} = 0.875$$

(3)⇒ $$J_{21} = \frac{1}{4}[-(1-0.5)4 - (1+0.5)7 + (1+0.5)8 + (1-0.5)3]$$

$$J_{21} = 0.25$$

(4)⇒ $$J_{22} = \frac{1}{4}[-(1-0.5)\times 4 - (1+0.5)\times 5 + (1+0.5)10 + (1-0.5)8]$$

$$J_{22} = 2.375$$

⇒ $$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 2.25 & 0.875 \\ 0.25 & 2.375 \end{bmatrix}$$

Result: Jacobian matrix, $[J] = \begin{bmatrix} 2.25 & 0.875 \\ 0.25 & 2.375 \end{bmatrix}$

Example 4.9

A four noded rectangular element is shown in Fig.(i). Determine the following

1. Jacobian Matrix 2. Strain- Dsiplacement matrix 3. Element Stresses.

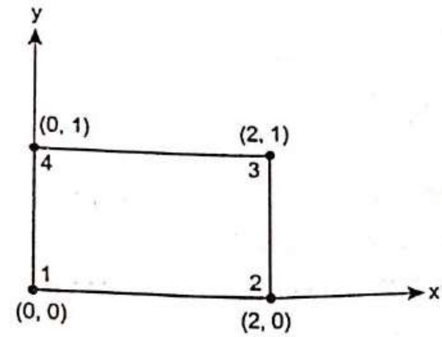


Fig. (i)

$$Take\ E = 2\ \times 10^5 N/mm^2 v = 0.25; u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0.0\ ]^T \varepsilon = 0;\ \eta = 0$$

Assume plane stress condition.

Given:

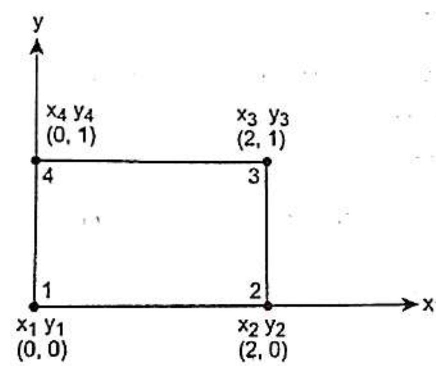


Fig. (ii)

Cartesian co-ordinates of point 1, 2, 3 and 4,

$$x_1 = 0; \qquad y_1 = 0$$

$$x_2 = 2; \qquad y_2 = 0$$

$$x_3 = 2; \qquad y_3 = 1$$

$$x_4 = 0; \qquad y_4 = 1$$

Young's modulus, $E = 2 \times 10^5 N/m^2$

Poisson's ratio, v = 0.25

Displacements, $u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$

Natural co-ordinates, ε = 0 , η = 0

To find:

1. Jacobian matrix, J.
2. Strain –Displacement matrix, [B].
3. Element stress, σ.

Solution: Jacodian matrix for quadrilateral element is given by,

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \quad \text{... (1)}$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \quad \text{... (2)}$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4] \quad \text{... (3)}$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4] \quad \text{... (4)}$$

Subsitutute $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, ε and η values in equaiton (1), (2), (3)and (4).

(1)⇒ $$J_{11} = \frac{1}{4}[0 + 2 + 2 - 0]$$

$$J_{11} = 1$$

(2)⇒ $J_{12} = \frac{1}{4}[0 + 0 + 1 - 1]$

$J_{12} = 0$

(3)⇒ $J_{21} = \frac{1}{4}[0 - 2 + 2 + 0]$

$J_{21} = 0$

(4)⇒ $J_{22} = \frac{1}{4}[- 0 - 0 + 1 + 1]$

$J_{22} = 0.5$

$$\Rightarrow \qquad [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Jacobian matrix, $[J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ ... (5)

$$\Rightarrow \qquad |J| = 1 \times 0.5 - 0$$

$|J| = 0.5$ ... (6)

$$\Rightarrow \qquad [B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\times \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

Substitute $J_{11}, J_{12}, J_{21}, J_{22}, |J|, \eta$ *and* ε *values,*

$$\Rightarrow \qquad [B] = \frac{1}{0.5}\begin{bmatrix} 0.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{0.5 \times 4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$= \frac{1}{0.5 \times 4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[B] = 0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix} \quad \text{... (7)}$$

We know that,

Element stress, $\sigma = [D][B]\{u\}$ ... (8)

For plane stress condition,

Stress – strain relationship matrix,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.25}{2} \end{bmatrix}$$

$$= 213.33 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 213.33 \times 10^3 \times 0.25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \quad \text{... (9)}$$

Substitute [D], [B] and { u } values in equation (8),

$$\Rightarrow \quad \{\sigma\} = 53.333 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\times 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 53.333 \times 10^3$$

$$\times 0.25 \begin{bmatrix} -4 & -2 & 4 & -2 & 4 & 2 & -4 & 2 \\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8 \\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 13.333$$

$$\times 10^3 \begin{Bmatrix} 0 + 0 + (4 \times 0.003) + (-2 \times 0.004) + (4 \times 0.006) + (2 \times 0.004) + 0 + 0 \\ 0 + 0 + (1 \times 0.003) + (-8 \times 0.004) + (1 \times 0.006) + (8 \times 0.004) + 0 + 0 \\ 0 + 0 + (-3 \times 0.003) + (1.5 \times 0.004) + (3 \times 0.006) + (1.5 \times 0.004) + 0 + 0 \end{Bmatrix}$$

$$\{\sigma\} = 13.333 \times 10^3 \begin{Bmatrix} 0.036 \\ 0.009 \\ 0.021 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 480 \\ 120 \\ 280 \end{Bmatrix} N/m^2$$

Result:

1. Jacobian matrix, $[J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$

2. Strain – Displacement matrix, $[B] = 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$

3. $Stress, \{\sigma\} = \begin{Bmatrix} 480 \\ 120 \\ 280 \end{Bmatrix} N/m^2$

Example:4.10

Consider a quadrilateral element as shown in fig.(i). The local co-ordinates are ϵ=0.5 and ή=1/2. Evaluate Jacobian Matrix and Strain – Displacement Matrix.

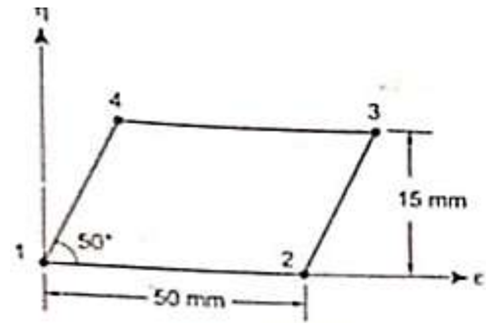


Fig. (i)

Given: Local co-ordinates

$\varepsilon = 0.5$

$\eta = 0.5$

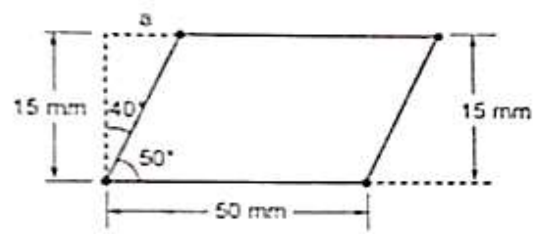


Fig. (ii)

$$tan\ 40^0 = \frac{a}{15}$$

$\Rightarrow$ a = 12.586 mm

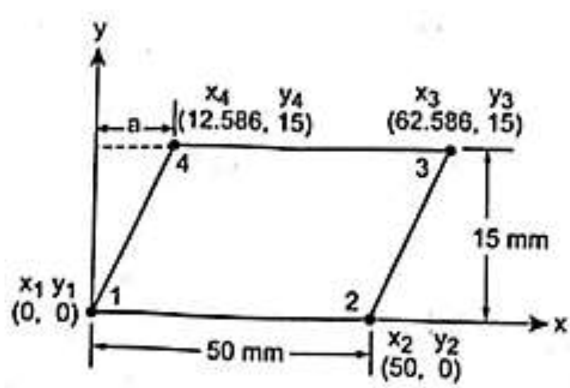


Fig. (iii)

The cartesian co-ordinates are:

$x_1 = 0;$ $\quad y_1 = 0$

$x_2 = 50\ mm;$ $\quad y_2 = 0$

$x_3 = 50 + 12.536 = 62.586\ mm;$ $\quad y_3 = 15\ mm$

$x_4 = a = 12.586\ mm;$ $\quad y_4 = 15\ mm$

To find:

1. Jacobian matrix, [J].
2. Strain –Displacement matrix, [B].

Solution: Jacodian matrix for quadrilateral element is given by,

$$[\,J\,] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \qquad ...(1)$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \qquad ...(2)$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4] \qquad ...(3)$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4] \qquad ...(4)$$

Subsitutute $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, ε and η values in equaiton (1), (2), (3)and (4).

(1)⇒ $$J_{11} = \frac{1}{4}[-(1-0.5)(0) + (1-0.5)\times 50 + (1+0.5)\times 62.586 - (1+0.5)12.586]$$

$$J_{11} = 25$$

(2)⇒ $$J_{12} = \frac{1}{4}[0 + 0 + (1+0.5)15 - (1+0.5)15]$$

$$J_{12} = 0$$

(3)⇒ $$J_{21} = \frac{1}{4}[-(1-0.5)\times 0 - (1+0.5)\times 50 + (1+0.5)\times 62.586 + (1-0.5)\times 12.586\,]$$

$$J_{21} = 6.293$$

(4)$\Rightarrow$ $$J_{22} = \frac{1}{4}[-0 - 0 + (1+0.5)15 + (1-0.5)15]$$

$$J_{22} = 7.5$$

$\Rightarrow$ $$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Jacobian matrix, $$[J] = \begin{bmatrix} 25 & 0 \\ 6.293 & 75 \end{bmatrix} \quad \ldots (5)$$

$\Rightarrow$ $$|J| = (25 \times 7.5) - (0 \times 6.293)$$

$$|J| = 187.5$$

We know that,

Strain –Displacement matrix for quadrilateral element is,

$\Rightarrow$ $$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\times \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

Substitute $J_{11}, J_{12}, J_{21}, J_{22}, |J|, \eta$ *and* ε *values,*

$\Rightarrow$ $$[B] = \frac{1}{187.5}\begin{bmatrix} 7.5 & 0 & 0 & 0 \\ 0 & 0 & -6.293 & 25 \\ -6.293 & 25 & 7.5 & 0 \end{bmatrix}$$

$$\times \frac{1}{4}\begin{bmatrix} -(1-0.5) & 0 & (1-0.5) & 0 & (1+0.5) & 0 & -(1+0.5) & 0 \\ -(1-0.5) & 0 & -(1+0.5) & 0 & (1+0.5) & 0 & (1-0.5) & 0 \\ 0 & -(1-0.5) & 0 & (1-0.5) & 0 & (1+0.5) & 0 & -(1+0.5) \\ 0 & -(1-0.5) & 0 & -(1+0.5) & 0 & (1+0.5) & 0 & (1-0.5) \end{bmatrix}$$

$$= \frac{1}{187.5 \times 4}\begin{bmatrix} 7.5 & 0 & 0 & 0 \\ 0 & 0 & -6.293 & 25 \\ -6.293 & 25 & 7.5 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 1.5 & 0 & -1.5 & 0 \\ -0.5 & 0 & -1.5 & 0 & 1.5 & 0 & 0.5 & 0 \\ 0 & -0.5 & 0 & 0.5 & 0 & 1.5 & 0 & -1.5 \\ 0 & -0.5 & 0 & -1.5 & 0 & 1.5 & 0 & 0.5 \end{bmatrix}$$

$$= \frac{1}{750}$$

$$\times \begin{bmatrix} 7.5 \times -(0.5) & 0 & 7.5 \times 0.5 & 0 & 7.5 \times 1.5 & 0 & 7.5 \times -(1.5) & 0 \\ 0 & \begin{pmatrix} -6.293 \times (-0.5) \\ + \\ 25 \times (-0.5) \end{pmatrix} & 0 & \begin{pmatrix} -6.293 \times 0.5 \\ + \\ 25 \times -1.5 \end{pmatrix} & 0 & \begin{pmatrix} -6.293 \times 1.5 \\ + \\ 25 \times 1.5 \end{pmatrix} & 0 & \begin{pmatrix} -6.293 \times -1.5 \\ + \\ 25 \times 1.5 \end{pmatrix} \\ \begin{pmatrix} -6.293 \times (-0.5) \\ + \\ 25 \times (-0.5) \end{pmatrix} & 7.5 \times -0.5 & \begin{pmatrix} -6.293 \times 0.5 \\ + \\ 25 \times -1.5 \end{pmatrix} & 7.5 \times 0.5 & \begin{pmatrix} -6.293 \times 1.5 \\ + \\ 25 \times 1.5 \end{pmatrix} & 7.5 \times 1.5 & \begin{pmatrix} -6.293 \times -1.5 \\ + \\ 25 \times 1.5 \end{pmatrix} & 7.5 \times -1.5 \end{bmatrix}$$

$$= \frac{1}{750}$$

$$\times \begin{bmatrix} -3.75 & 0 & 3.75 & 0 & 11.25 & 0 & -11.25 & 0 \\ 0 & -9.3535 & 0 & -40.6465 & 0 & 28.0605 & 0 & 21.939 \\ -9.3535 & -3.75 & -40.6465 & 3.75 & 28.0605 & 7.875 & 21.939 & -7.875 \end{bmatrix}$$

$$[B] = \frac{1}{750}$$

$$\times \begin{bmatrix} -3.75 & 0 & 3.75 & 0 & 11.25 & 0 & -11.25 & 0 \\ 0 & -9.3535 & 0 & -40.6465 & 0 & 28.0605 & 0 & 21.939 \\ -9.3535 & -3.75 & -40.6465 & 3.75 & 19.6423 & 7.875 & 21.939 & -7.875 \end{bmatrix}$$

Result:

1. Jacobian matrix, $[\,J\,] = \begin{bmatrix} 25 & 0 \\ 6.293 & 7.5 \end{bmatrix}$

2. Strain − Displacement matrix,

$$[\mathrm{B}] = \frac{1}{750}$$

$$\times \begin{bmatrix} -3.75 & 0 & 3.75 & 0 & 11.25 & 0 & -11.25 & 0 \\ 0 & -9.3535 & 0 & -40.6465 & 0 & 28.0605 & 0 & 21.939 \\ -9.3535 & -3.75 & -40.6465 & 3.75 & 19.6423 & 7.875 & 21.939 & -7.875 \end{bmatrix}$$

Example 4.11

Establish the strain-displacement matrix for the linear quadrilateral element as shown in Fig.(i) at Gauss point r = 0.57735 and s = 0.57735.

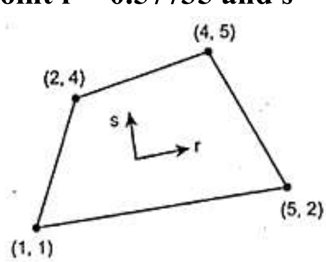


Fig. (i)

Given:

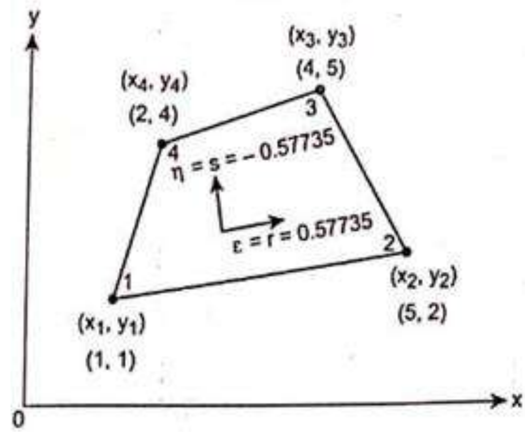


Fig. (ii)

Cartesian co-ordinates of point 1, 2, 3 and 4 are:

$x_1 = 1;$ $\quad$ $y_1 = 1$

$x_2 = 5;$ $\quad$ $y_2 = 2$

$x_3 = 4;$ $\quad$ $y_3 = 5$

$x_4 = 2;$ $\quad$ $y_4 = 4$

Natural co-ordinates, ε = 0.57735

η = -0.57735

To find: Strain –Displacement matrix, [B]

Solution: Jacodian matrix for quadrilateral element is given by,

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \quad ...(1)$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \quad ...(2)$$

$$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4] \quad ...(3)$$

$$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4] \quad ...(4)$$

Subsitutute $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, ε and η values in equaiton (1), (2), (3)and (4).

(1)⇒ $$J_{11} = \frac{1}{4}[-(1+0.57735)(1) + (1+0.57735)(5) + (1-0.57735)(4) - (1-0.57735)(2)]$$

$$J_{11} = 1.78867$$

(2)⇒ $$J_{12} = \frac{1}{4}[-(1+0.57735)(1) + (1+0.57735)(2) + (1-0.57735)(5) - (1-0.57735)(4)]$$

$$J_{12} = 0.5$$

(3)⇒ $$J_{21} = \frac{1}{4}[-(1-0.57735)(1) - (1+0.57735)(5) + (1+0.57735)(4) + (1-0.57735)(2)]$$

$$J_{21} = -0.2886$$

(4)⇒ $$J_{22} = \frac{1}{4}[-(1-0.57735)(1) - (1+0.57735)(2) + (1+0.57735)(5) + (1-0.57735)(4)]$$

$$J_{22} = 1.5$$

⇒ $$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$|J| = \begin{vmatrix} 1.7886 & 0.5 \\ -0.2886 & 1.5 \end{vmatrix}$$

$$|J| = 2.827$$

We know that,

Strain –Displacement matrix for quadrilateral element is,

$$\Rightarrow \qquad [\,B\,] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\times \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

Substitute $J_{11}, J_{12}, J_{21}, J_{22}, |J|, \eta\ and\ \varepsilon\ values,$

$$\Rightarrow \qquad [\,B\,] = \frac{1}{2.827}\begin{bmatrix} 1.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0.2886 & 1.7886 \\ 0.2886 & 1.7886 & 1.5 & -0.5 \end{bmatrix}$$

$$\times \frac{1}{4}\begin{bmatrix} -1.57735 & 0 & 1.57735 & 0 & 0.42265 & 0 & -0.42265 & 0 \\ -0.42265 & 0 & -1.57735 & 0 & 1.57735 & 0 & 0.42265 & 0 \\ 0 & -1.57735 & 0 & 1.57735 & 0 & 1.57735 & 0 & -0.42265 \\ 0 & -0.42265 & 0 & -1.57735 & 0 & & 0 & 0.42265 \end{bmatrix}$$

$$= \frac{1}{2.827 \times 4}\begin{bmatrix} -2.1547 & 0 & 3.1547 & 0 & -0.1547 & 0 & -0.8453 & 0 \\ 0 & -1.2111 & 0 & -2.3660 & 0 & 2.9432 & 0 & 0.63397 \\ -1.2111 & -2.1547 & -2.3660 & 3.1457 & 2.9432 & -0.1547 & 0.63397 & -0.8453 \end{bmatrix}$$

$$[B] = 0.0884\begin{bmatrix} -2.1547 & 0 & 3.1547 & 0 & -0.1547 & 0 & -0.8453 & 0 \\ 0 & -1.2111 & 0 & -2.3660 & 0 & 2.9432 & 0 & 0.63397 \\ -1.2111 & -2.1547 & -2.3660 & 3.1457 & 2.9432 & -0.1547 & 0.63397 & -0.8453 \end{bmatrix}$$

Result:

1. $Strain - Displacement\ matrix, [B]$

$$= 0.0884\begin{bmatrix} -2.1547 & 0 & 3.1547 & 0 & -0.1547 & 0 & -0.8453 & 0 \\ 0 & -1.2111 & 0 & -2.3660 & 0 & 2.9432 & 0 & 0.63397 \\ -1.2111 & -2.1547 & -2.3660 & 3.1457 & 2.9432 & -0.1547 & 0.63397 & -0.8453 \end{bmatrix}$$

Example 4.12

For the isoparametric quadrilateral element shown in Fig. (i), the Cartesian co-ordinates of point P are (6,4). The loads 10 kN and 12 kN are acting in x and y directions on that point P. Evaluate the nodal equivalent forces.

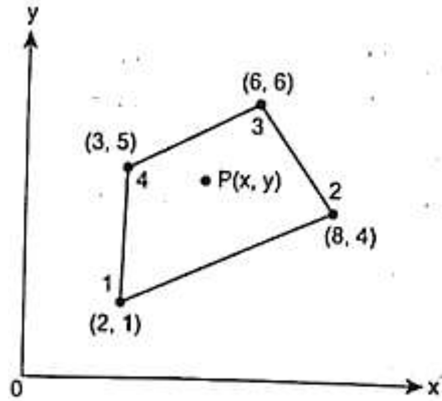


Fig. (i)

Given: Cartesian co-ordinates of point P,

$x = 6;$ $y = 4$

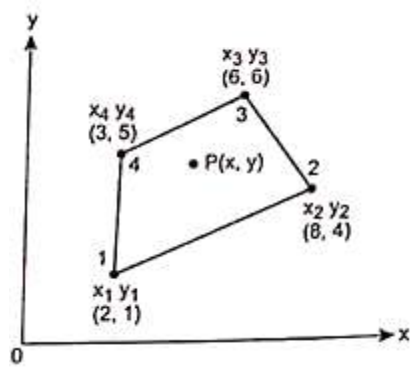


Fig. (ii)

The Cartesian co-ordinates of point 1, 2, 3 and 4 are:

$x_1 = 2;$ $y_1 = 1$

$x_2 = 8;$ $y_2 = 4$

$x_3 = 6;$ $y_3 = 6$

$x_4 = 3;$ $y_4 = 5$

Loads, $F_x = 10\ kN,$ $F_y = 12\ kN$

To find: Nodal equivalent forces for x and y directions,

$i.e., F_{1x}, F_{2x}, F_{3x}, F_{4x}, F_{1y}, F_{2y}, F_{3y}, F_{4y},$

Solution: we know that,

Shape functions for quadrilateral element are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \qquad \ldots (1)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta) \qquad \ldots (2)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \qquad \ldots (3)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta) \qquad \ldots (4)$$

Cartesian co-ordinates of point, P(x, y).

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \qquad \ldots (5)$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \qquad \ldots (6)$$

Substitute $x, x_1, x_2, x_3, x_4, N_1, N_2, N_3$ and N_4 values in equation (5),

$$6 = \frac{1}{4}[(1-\varepsilon)(1-\eta)2 + (1+\varepsilon)(1-\eta)8 + (1+\varepsilon)(1+\eta)6 + (1-\varepsilon)(1+\eta)3]$$

$$24 = [(1-\eta-\varepsilon+\varepsilon\eta)2 + (1-\eta+\varepsilon-\varepsilon\eta)8 + (1+\eta+\varepsilon+\varepsilon\eta)6 + (1+\eta-\varepsilon-\varepsilon\eta)3]$$

$$24 = [2-2\eta-2\varepsilon+2\varepsilon\eta+8-8\eta+8\varepsilon-8\varepsilon\eta+6+6\eta+6\varepsilon+6\varepsilon\eta+3 + 3\eta+3\varepsilon-3\varepsilon\eta]$$

$$24 = 19-\eta+9\varepsilon-3\varepsilon\eta$$

$$5 = -\eta+9\varepsilon-3\varepsilon\eta$$

$$\Rightarrow \qquad 9\varepsilon-\eta-3\varepsilon\eta = 5 \qquad \ldots (7)$$

Substitute $y, y_1, y_2, y_3, y_4, N_1, N_2, N_3$ and N_4 values in equation (6),

$$4 = \frac{1}{4}[(1-\varepsilon)(1-\eta)1 + (1+\varepsilon)(1-\eta)4 + (1+\varepsilon)(1+\eta)6 + (1-\varepsilon)(1+\eta)5]$$

$$16 = [1 - \eta - \varepsilon + \varepsilon\eta + (1 - \eta + \varepsilon - \varepsilon\eta) \times 4 + (1 + \eta + \varepsilon + \varepsilon\eta) \times 6 + (1 + \eta - \varepsilon - \varepsilon\eta) \times 5]$$

$$= [1 - \eta - \varepsilon + \varepsilon\eta + 4 - 4\eta + 4\varepsilon - 4\varepsilon\eta + 6 + 6\eta + 6\varepsilon + 6\varepsilon\eta + 5 + 5\eta - 5\varepsilon - 5\varepsilon\eta]$$

$$16 = [16 + 6\,\eta + 4\varepsilon - 2\varepsilon\eta]$$

$$\Rightarrow \quad 4\varepsilon + 6\eta - 2\varepsilon\eta = 0 \qquad \ldots (8)$$

Equation (7) multiplied by 2 and equation (8) multiplied by (-3)

$$18\varepsilon - 2\,\eta - 6\varepsilon\eta = 10 \qquad \ldots (9)$$

$$-12\varepsilon - 18\eta + 6\varepsilon\eta = 0 \qquad \ldots (10)$$

Solving, $\quad 6\varepsilon - 20\eta = 0$

$$\Rightarrow \quad -20\eta = 10 - 6\varepsilon$$

$$\Rightarrow \quad 20\eta = 6\varepsilon - 10$$

$$\Rightarrow \quad \eta = \frac{6\varepsilon - 10}{20}$$

$$\Rightarrow \quad \eta = 0.3\varepsilon - 0.5 \qquad \ldots (11)$$

Substituting η values in equation (7),

$$\Rightarrow \quad 9\varepsilon - (0.3\varepsilon - 0.5) - 3\varepsilon(0.3\varepsilon - 0.5) = 5$$

$$\Rightarrow \quad 9\varepsilon - 0.3\varepsilon - 0.5 - 0.9\varepsilon^2 + 1.5\varepsilon = 5$$

$$\Rightarrow \quad 10.2\varepsilon - 0.9\varepsilon^2 - 4.5 = 0$$

$$\Rightarrow \quad 0.9\varepsilon^2 - 10.2\varepsilon + 4.5 = 0$$

$$\Rightarrow \qquad \varepsilon = \frac{10.2 \pm \sqrt{(-10.2)^2 - 4(0.9)(4.5)}}{2(0.9)}$$

$$= \frac{10.2 - 9.372}{1.8}$$

$$\varepsilon = 0.46$$

Substitute ε value in equation (11),

$$\Rightarrow \qquad \eta = 0.3(0.46)\text{-}0.5$$

$$\eta = \text{-}0.362$$

Substitute ε and η values in equation (1), (2), (3) and (4).

$$(1) \Rightarrow \qquad N_1 = \frac{1}{4}(1 - 0.46)(1 + 0.362)$$

$$N_1 = 0.18387$$

$$(2) \Rightarrow \qquad N_2 = \frac{1}{4}(1 + 0.46)(1 + 0.362)$$

$$N_2 = 0.49713$$

$$(3) \Rightarrow \qquad N_3 = \frac{1}{4}(1 + 0.46)(1 - 0.362)$$

$$N_3 = 0.23287$$

$$(4) \Rightarrow \qquad N_4 = \frac{1}{4}(1 - 0.46)(1 - 0.362)$$

$$N_4 = 0.08613$$

We know that,

Element force vector, $\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$ ... (12)

$$\Rightarrow \qquad \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3X} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \{F_x\}$$

$$\Rightarrow \qquad \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3X} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 0.18384 \\ 0.49713 \\ 0.23287 \\ 0.08613 \end{Bmatrix} \{10\}$$

$$\Rightarrow \qquad \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 0.18384 \\ 0.49713 \\ 0.23287 \\ 0.08613 \end{Bmatrix} kN$$

Similarly, $$\begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \{F_y\}$$

$$= \begin{Bmatrix} 0.18387 \\ 0.49713 \\ 0.23287 \\ 0.08613 \end{Bmatrix} \{12\}$$

$$\Rightarrow \qquad \begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} 2.20644 \\ 5.96556 \\ 2.79444 \\ 1.03356 \end{Bmatrix} kN$$

Result: Nodal forces for x directions,

$$\Rightarrow \qquad \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 1.8387 \\ 4.9713 \\ 2.3287 \\ 0.8613 \end{Bmatrix} kN$$

Nodal forces for y directions,

$$\Rightarrow \qquad \begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} 2.20644 \\ 5.96556 \\ 2.79444 \\ 1.03356 \end{Bmatrix} kN$$

Example 4.13

Evaluate the Jacobian matrix for the isoparametric quadrilateral element shown in Fig. (i).

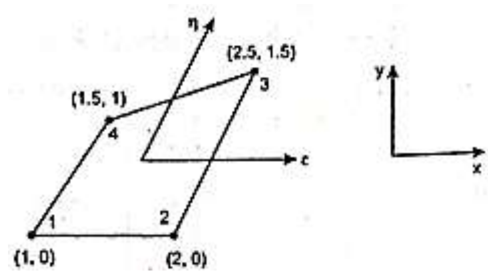


Fig. (i).

Given: Cartesian co-ordinates:

$x_1 = 1;$	$y_1 = 0$
$x_2 = 2;$	$y_2 = 0$
$x_3 = 2.5;$	$y_3 = 1.5$
$x_4 = 1.5;$	$y_4 = 1$

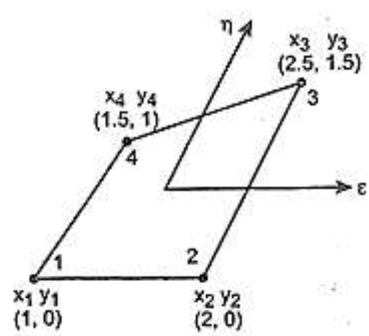


Fig. (ii).

To find: Jacobian matrix, [J]

Solution: Shape functions for isoparametric quadrilateral element are given by,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \qquad \dots (1)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta) \qquad \dots (2)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \qquad \text{... (3)}$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta) \qquad \text{... (4)}$$

We know that, for quadrilateral element, Cartesian co-ordinates are:

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \qquad \text{... (5)}$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \qquad \text{... (6)}$$

Substitute $N_1, N_2, N_3, N_4, x_1, x_2, x_3, x_4, y_1, y_2, y_3$ *and* y_4 values in equation (5) and (6).

$$x = \frac{1}{4}[(1-\varepsilon)(1-\eta)\times 1 + (1+\varepsilon)(1-\eta)\times 2 + (1+\varepsilon)(1+\eta)\times 2.5 + (1-\varepsilon)(1+\eta)\times 1.5] \qquad \text{...(7)}$$

$$y = \frac{1}{4}\ [(1-\varepsilon)(1-\eta)\times 0 + (1+\varepsilon)(1-\eta)\times 0 + (1+\varepsilon)(1+\eta)\times 1.5 + (1-\varepsilon)(1+\eta)\times 1] \qquad \text{...(8)}$$

Simplifying equation (7),

$$x = \frac{1}{4}\ [1+\ \eta - \varepsilon + \varepsilon\eta + 2 - 2\eta + 2\varepsilon - 2\varepsilon\eta + 2.5 + 2.5\,\eta + 2.5\varepsilon + 2\,.5\ \varepsilon\eta + 1.5 + 1.5\eta - 1.5\ \varepsilon - 1.5\ \varepsilon\eta]$$

$$= \frac{1}{4}\ [7+\ \eta + 2\varepsilon + 0\ \varepsilon\eta]$$

$$\Rightarrow \qquad x = \frac{1}{4}\ \ [7+\ \eta + 2\varepsilon] \qquad \text{... (9)}$$

$$\Rightarrow \qquad \frac{\partial x}{\partial \varepsilon} = \frac{1}{4}[0+0+2]$$

$$\frac{\partial x}{\partial \varepsilon} = \frac{1}{2} \qquad \text{... (10)}$$

$$\Rightarrow \qquad \frac{\partial x}{\partial \eta} = \frac{1}{4}[0+1+0]$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4} \qquad \text{... (11)}$$

Substitute equation (8),

$$y = \frac{1}{4}[0 + 0 + 1.5 + 1.5\,\eta + 1.5\,\varepsilon + 1.5\,\varepsilon\eta + 1 + \eta - \varepsilon - \varepsilon\eta]$$

$$y = \frac{1}{4}[2.5 + 2.5\,\eta + 0.5\,\varepsilon + 0.5\,\varepsilon\eta] \qquad \text{... (12)}$$

$$\Rightarrow \qquad \frac{\partial y}{\partial \varepsilon} = \frac{1}{4}[0 + 0 + 0.5 + 0.5\,\eta] = \frac{0.5}{4}[\,1 + \ \eta]$$

$$\frac{\partial y}{\partial \varepsilon} = 0.125[1 + \eta] \qquad \text{... (13)}$$

$$\Rightarrow \quad \frac{\partial y}{\partial \eta} = \frac{1}{4}[0 + 2.5 + 0.5\,\varepsilon] = \frac{0.5}{4}[\,5 + \ \varepsilon]$$

$$\frac{\partial y}{\partial \eta} = 0.125[5 + \varepsilon] \qquad \text{... (14)}$$

We know that, Jacobian matrix, $[\,J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

Where, $J_{11} = \frac{\partial x}{\partial \varepsilon}; \qquad J_{12} = \frac{\partial y}{\partial \varepsilon}$

$$J_{21} = \frac{\partial x}{\partial \eta}; \qquad J_{22} = \frac{\partial y}{\partial \eta}$$

$$\Rightarrow \qquad [\,J] = \begin{bmatrix} \frac{1}{2} & 0.125(1 + \eta) \\ \frac{1}{4} & 0.125(1 + \varepsilon) \end{bmatrix}$$

Result: Jacobian matrix, $[\,J] = \begin{bmatrix} \frac{1}{2} & 0.125(1 + \eta) \\ \frac{1}{4} & 0.125(1 + \varepsilon) \end{bmatrix}$

Example 4.14

For the element shown in Fig. (i), determine the Jacobian matrix

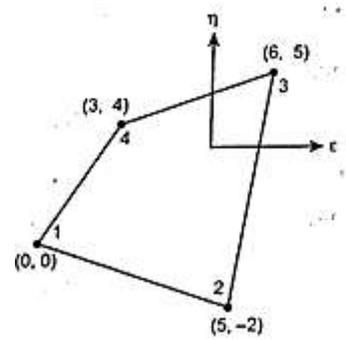


Fig. (i)

Given: Cartesian co-ordinates:

$x_1 = 0;$ $\qquad y_1 = 0$

$x_2 = 5;$ $\qquad y_2 = -2$

$x_3 = 6;$ $\qquad y_3 = 5$

$x_4 = 3;$ $\qquad y_4 = 4$

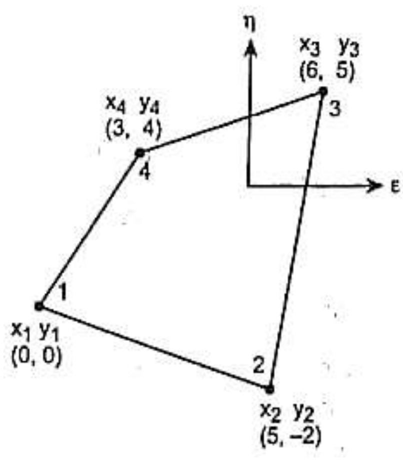


Fig. (ii).

To find: Jacobian matrix, [J]

Solution: Shape functions for isoparametric quadrilateral element are given by,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \qquad \text{... (1)}$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta) \qquad \text{... (2)}$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \qquad \text{... (3)}$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta) \qquad \text{... (4)}$$

We know that, for quadrilateral element, Cartesian co-ordinates are:

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \qquad \text{... (5)}$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \qquad \text{... (6)}$$

Substitute $N_1, N_2, N_3, N_4, x_1, x_2, x_3, x_4, y_1, y_2, y_3$ and y_4 values in equation (5) and (6).

$$x = \frac{1}{4}[(1-\varepsilon)(1-\eta) \times 0 + (1+\varepsilon)(1-\eta) \times 5 + (1+\varepsilon)(1+\eta) \times 6$$
$$+ (1-\varepsilon)(1+\eta) \times 3] \qquad \ldots(7)$$

$$y = \frac{1}{4}\ [(1-\varepsilon)(1-\eta) \times 0 + (1+\varepsilon)(1-\eta) \times (-2)$$
$$+ (1+\varepsilon)(1+\eta) \times 5 + (1-\varepsilon)(1+\eta) \times 4] \qquad \ldots(8)$$

Simplifying equation (7),

$$x = \frac{1}{4}\ [5 - 5\eta + 5\varepsilon - 5\varepsilon\eta + 6 + 6\eta + 6\varepsilon + 6\varepsilon\eta + 3 + 3\eta - 3\varepsilon - 3\varepsilon\eta]$$

$$x = \frac{1}{4}\ [14 + \ 4\eta + 8\varepsilon - 2\ \varepsilon\eta] \qquad \text{... (9)}$$

$$\Rightarrow \qquad \frac{\partial x}{\partial \varepsilon} = \frac{1}{4}[0 + 0 + 8 - 2\eta] = \frac{1}{4}[\ 8 - 2\eta] = \frac{4}{4}[2 - 0.5\eta]$$

$$\frac{\partial x}{\partial \varepsilon} = 2 - 0.5\eta \qquad \text{... (10)}$$

$$\Rightarrow \qquad \frac{\partial x}{\partial \eta} = \frac{1}{4}[0 + 4 + 0 - 2\varepsilon] = \frac{4}{4}[1 - 0.5\,\varepsilon]$$

$$\frac{\partial x}{\partial \eta} = 1 - 0.5\,\varepsilon \qquad \ldots (11)$$

Substitute equation (8),

$$y = \frac{1}{4}[+0 - 2 + 2\eta - 2\varepsilon + 2\varepsilon\eta + 5 + 5\eta + 5\varepsilon + 5\varepsilon\eta + 4 + 4\eta - 4\varepsilon - 4\varepsilon\eta]$$

$$y = \frac{1}{4}[7 + 11\,\eta - \varepsilon + 3\,\varepsilon\eta] \qquad \ldots (12)$$

$$\Rightarrow \qquad \frac{\partial y}{\partial \varepsilon} = \frac{1}{4}[0 + 0 - 1 + 3\,\eta]$$

$$\frac{\partial y}{\partial \varepsilon} = 0.75\eta - 0.25 \qquad \ldots (13)$$

$$\Rightarrow \qquad \frac{\partial y}{\partial \eta} = \frac{1}{4}[0 + 11 - 0 + 3\,\varepsilon]$$

$$\frac{\partial y}{\partial \eta} = 2.75 + 0.75\,\varepsilon \qquad \ldots (14)$$

We know that,

Jacobian matrix, $[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

Where, $J_{11} = \frac{\partial x}{\partial \varepsilon}$; $\quad J_{12} = \frac{\partial y}{\partial \varepsilon}$

$J_{21} = \frac{\partial x}{\partial \eta}$; $\quad J_{22} = \frac{\partial y}{\partial \eta}$

$$\Rightarrow \; [J] = \begin{bmatrix} 2 - 0.5\eta & 0.75\eta - 0.25 \\ 1 - 0.5\varepsilon & 2.75 + 0.75\varepsilon \end{bmatrix}$$

Result: Jacobian matrix, $[J] = \begin{bmatrix} 2 - 0.5\eta & 0.75\eta - 0.25 \\ 1 - 0.5\varepsilon & 2.75 + 0.75\varepsilon \end{bmatrix}$

4.11. NUMERICAL INTEGRATION [GAUSSIAN QUADRATURE] AND APPLICATION TO PLANE STRESS PROBLEMS

The Gauss quadrature is one of the numerical integration methods to calculate the definite integrals. In finite element analysis, Gauss quadrature method is mostly preferred. In this method, the numerical integration is achieved by the following expression,

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i\, f(x)$$

where, $w_i$is weight function.

$f(x_i)$ is values of the function at pre-determined sampling points.

Function $f(x_i)$ is calculated at several sampling points i.e. n = 1, 2, 3,… and each value of $f(x_i)$ is multiplied by weight function w_iFinally, all the terms are added, it gives the value of integration.

Table 4.1 shows the location of Gauss sampling points $f(x_i)$ and corresponding weight function w for different number of points (n).

Table 4.1 gives Gauss points for integration from -1 to +1,

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i\, f(x_i)$$

Table 4.1.

Number of points, n	**Location, . x_i**	**Corresponding weights, w_i**
1.	$x_1 = 0.000$	2.000
2.	$x_{1,}x_2 = \pm\sqrt{\frac{1}{3}} = \pm 0.577350269189$	1.000
3.	$x_{1,}x_3 = \pm\sqrt{\frac{3}{5}} = \pm 0.774596669241$	$\frac{5}{9} = 0.555555\ldots$

	$x_2 = 0.000$	$\frac{8}{9} = 0.888888 \ldots$
4.	$x_1, x_4 = \pm 0.8611363116$ $x_2, x_3 = \pm 0.3399810436$	0.3478548451 0.6521451549

4.12. SOLVED PROBLEMS-GAUSSIAN QUADRATURE

Example 4.16

Evaluate $\int_{-1}^{1}(x^4 + x^2)\,dx$ by applying 3 point gaussian quadrature

Given

Integral $$I = \int_{-1}^{1} (x^4 + x^2)\,dx$$

$$f(x) = x^4 + x^2$$

To find: Evaluate the integral by using Gaussian quadrature with three Gauss points.

Solution: We know that, for three point Gaussian quadrature

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \frac{5}{9} = 0.555555$$

[Refer Table 4.1]

We know that, $f(x) = x^4 + x^2$

$$f(x_1) = (x_1)^4 + (x_1)^2$$

$$= (0.774596669)^4 + (0.774596669)^2$$

$$f(x_1) = 0.96$$

$$w_1\, f(x_1) = 0.555555 \times 0.96$$

$$w_1\, f(x_1) = 0.5333 \qquad \text{... (1)}$$

$$f(x_2) = (x_2)^4 + (x_2)^2 = (0)^4 + (0)2$$

$$f(x_2) = 0$$

$$w_2\, f(x_2) = 0.888888 \times 0$$

$$w_2\, f(x_2) = 0 \qquad \text{... (2)}$$

$$f(x_3) = (x_3)^4 + (x_3)^2$$

$$= (-0.774596669) + (-0.774596669)^2$$

$$f(x_3) = 0.96$$

$$w_3\, f(x_3) = 0.555555 \times 0.96$$

$$w_3\, f(x_3) = 0.5333 \qquad \text{... (3)}$$

Adding equation (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3) = 0.5333 + 0 + 0.5333 = 1.0666$$

Result:

$$\int_{-1}^{1} (x^4 + x^2)\, dx = 1.0666$$

Verification:

$$\int_{-1}^{1} (x^4 + x^2)\, dx = \left[\frac{x^5}{5}\right]_{-1}^{+1} + \left[\frac{x^3}{3}\right]_{-1}^{+1}$$

$$= \frac{1}{5}\,[(1^5) - (-1^5)] + \frac{1}{3}\,[(1^3) - (-1^3)]$$

$$= \frac{1}{5}[1 + 1] \; + \; \frac{1}{3}[1 + 1] = 1.0666$$

Example 4.17

Evaluate the integral $\displaystyle\int_{-1}^{1} (\mathbf{x^4 - 3x + 7})\,\mathbf{dx}$

Given:

Integral $\qquad \mathrm{I} = \displaystyle\int_{-1}^{1} (x^4 - 3x + 7)\,dx$

$$f(x) = (x^4 - 3x + 7)$$

To find: Evaluate the integral by using Gaussian quadrature.

Solution: We know that, the given integrand is a polynomial of order 2. So for exact integration,

$$2n - 1 = 4$$

$$2n = 5$$

$$n = \frac{5}{2} = 2.5$$

The calculated number of sampling points should be rounded upto the nearest integer value. So , n =2.5≈3, i.e., in this problem, we should use three sampling points.

For three point Gaussian quadrature,

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \frac{5}{9} = 0.555555 \qquad \text{[Refer Table 4.1]}$$

We know that,

$$f(x) = (x^4 - 3x + 7)$$

$$f(x_1) = ({x_1}^4 - 3x_1 + 7)$$

$$= (0.774596669)^4 - 3(0.774596669) + 7^4$$

$$f(x_1) = 5.036209992$$

$$w_1\, f(x_1) = 0.555555 \times 5.036209992$$

$$w_1 \times f(x_1) = 2.797891$$

$$f(x_2) = ({x_2}^4 - 3x_2 + 7) = (0)4 - 3(0) + 7$$

$$f(x_2) = 7$$

$$w_2\, f(x_2) = 0.888888 \times 7$$

$$w_2\, f(x_2) = 6.222216$$

$$f(x_3) = ({x_3}^4 - 3x_3 + 7)$$

$$= (-0.774596669)^4 - 3(-0.774596669) + 7$$

$$f(x_3) = 9.683790008$$

$$w_3\, f(x_3) = 0.555555 \times 9.683790008$$

$$w_3\, f(x_3) = 5.379877$$

Adding equation (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3)$$
$$= 2.797891 + 6.222216 + 5.379877$$
$$= 14.399984$$

Result:

$$\int_{-1}^{1} (x^4 - 3x + 7)\, dx = 14.399984$$

Verification:

$$\int_{-1}^{1} (x^4 - 3x + 7)\, dx = \left[\frac{x^5}{5}\right]_{-1}^{+1} - 3\left[\frac{x^2}{2}\right]_{-1}^{+1} + 7[x]_{-1}^{+1}$$
$$= \frac{1}{5}\,[(1^5) - (-1^5)] - \frac{3}{2}[(1^2) - (-1^2)] + 7\,[\,1 - (1)]$$
$$= \frac{1}{5}[2] - 0 + 7(2) = 14.4$$

Example 4.18

Evaluate the integral by gaussian quadrature $\int_{-1}^{1} \mathbf{x^2 dx}$

Given:

Integral, $\quad \mathrm{I} = \int_{-1}^{1} \boldsymbol{x^2 dx}$

$$f(x) = x^2$$

To find: Evaluate the integral by using Gaussian quadrature.

Solution: We know that, the given integrand is a polynomial of order 2. So for exact integration,

$$2n - 1 = 2$$
$$2n = 3$$

$$n = \frac{3}{2} = 1.5$$

The calculated number of sampling points should be rounded upto the nearest integer value. So , n =1.5≈2, i.e., in this problem, we should use two sampling points.

For three point Gaussian quadrature,

$$x_1 = +\sqrt{\frac{1}{3}} = 0.577350269$$

$$x_2 = -\sqrt{\frac{1}{3}} = -0.577350269$$

$$w_1 = 1$$

$$w_2 = 1 \qquad \text{[Refer Table 4.1]}$$

We know that,

$$f(x) = x^2$$

$$f(x_1) = {x_1}^2 = (0.577350269)^2$$

$$f(x_1) = 0.333333333$$

$$w_1\, f(x_1 = 1 \times 0.333333333)$$

$$w_1\, f(x_1) = 0.333333333 \qquad \ldots.(1)$$

$$f(x_2) = {x_2}^2 = (-0.577350269)^2$$

$$f(x_2) = 0.333333333$$

$$w_2\, f(x_2) = 1 \times 0.333333333$$

$$w_2\, f(x_2) = 0.333333333 \qquad \ldots.(2)$$

Adding (1),(2)

$$w_1\, f(x_1) + w_2\, f(x_2) = 0.333333333 + 0.333333333 = 0.666666666$$

Result:

$$\int_{-1}^{1} x^2 dx = 0.666666666$$

Example 4.19

Evaluate the integral I $= \int_{-1}^{1}(2 + x + x^2)$ dx and compare with exact solution.

Given:

$$\text{Integral, I} = \int_{-1}^{1} (\mathbf{2} + \boldsymbol{x} + \boldsymbol{x}^{\mathbf{2}})\, \boldsymbol{dx}$$

To find: Evaluate the integral by using Gaussian quadrature.

Solution: We know that, the given integrand is a polynomial of order 2.

So, $\quad 2n - 1 = 2$

$2n = 3$

$n = 1.5 \approx 2$

We should use two sampling points.

For two point Gaussian quadrature,

$$x_1 = +\sqrt{\frac{1}{3}} = 0.577350269$$

$$x_2 = -\sqrt{\frac{1}{3}} = -0.577350269$$

$$w_1 = 1$$

$$w_2 = 1 \qquad \text{[Refer Table 4.1]}$$

We know that,

$$f(x) = 2 + x + x^2$$

$$f(x_1) = 2 + x_1 + {x_1}^2$$

$$= 2 + (0.577350269) + (0.577350269)^2$$

$$f(x_1) = 2.9106836$$

$$w_1\, f(x_1) = 1 \times 2.9106836$$

$$w_1\, f(x_1) = 2.9106836 \qquad \text{....(1)}$$

$$f(x_2) = 2 + x_2 + {x_2}^2$$

$$= 2 - (0.577350269) + (-0.577350269)^2$$

$$f(x_2) = 1.755983$$

$$w_2\, f(x_2) = 1 \times 1.755983$$

$$w_2\, f(x_2) = 1.755983 \qquad \text{....(2)}$$

Adding (1) & (2),

$$w_1\, f(x_1) + w_2\, f(x_2) = 2.9106836 + 1.755983 = 4.666666$$

$$\int_{-1}^{1} (2 + x + x^2)\, dx = 4.666666$$

Exact solution:

$$\int_{-1}^{1} (2 + x + x^2)\, dx = 2[x]_{-1}^{+1} + \frac{1}{2}[x^2]_{-1}^{+1} + \frac{1}{3}[x^2]_{-1}^{+1}$$

$$= 2[1 - (1)] + \frac{1}{2}[1 - (1)] + \frac{1}{3}[1 - (1)]$$

$$= 4.666666$$

Result:

$$\int_{-1}^{1} (2 + x + x^2)\, dx = 4.666666 \qquad \text{[By two point Gaussian quadrature]}$$

$$\int_{-1}^{1} (2 + x + x^2)\, dx = 4.666666 \qquad \text{[By exact method]}$$

Example 4.20

Evaluate $\int_{-1}^{1} e^{-x}\,dx$ by applying 3 point gaussian quadrature.

Given:

Integral, $$I = \int_{-1}^{1} e^{-x}\,dx$$

$$f(x) = e^{-x}\,dx$$

To find: Evaluate the integral by using Gaussian quadrature.

Solution: We know that, for three point Gaussian quadrature

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \frac{5}{9} = 0.555555$$ [Refer Table 4.1]

We know that,

$$f(x) = e^{-x}$$

$$f(x_1) = e^{-x_1} = e^{-0.774596669}$$

$$f(x_1) = 0.460889634$$

$$w_1\, f(x_1) = 0.555555 \times 0.460889634$$

$$w_1\, f(x_1) = 0.25604954 \qquad \ldots.(1)$$

$$f(x_2) = e^{-x_2} = e^{-0}$$

$$f(x_2) = 1$$

$$w_2\, f(x_2) = 0.888888 \times 1$$

$$w_2\, f(x_2) = 0.888888 \qquad \ldots.(2)$$

$$f(x_3) = e^{-x_3} = e^{-(-0.774596669)}$$

$$f(x_3) = e^{0.774596669}$$

$$f(x_3) = 2.169716837$$

$$w_3\, f(x_3) = 0.555555 \times 2.169716837$$

$$w_3\, f(x_3) = 1.205397037 \qquad \ldots.(3)$$

Adding equations (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3)$$
$$= 0.25604954 + 0.888888 + 1.205397037$$
$$= 2.350334$$

Result:

$$\int_{-1}^{1} e^{-x}\, dx = 2.350334$$

Example 4.21

Evaluate $\int_{-1}^{1} \cos\frac{x}{2}\, dx$ by applying 3 point gaussian quadrature.

Given:

Integral, $\quad \mathrm{I} = \int_{-1}^{1} \cos\frac{x}{2}\, dx$

$$f(x) = \cos\frac{x}{2}$$

To find: Evaluate the integral by using Gaussian quadrature.

Solution: We know that, for three point Gaussian quadrature

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \frac{5}{9} = 0.555555$$ [Refer Table 4.1]

We know that,

$$f(x) = \cos\frac{x}{2}$$

$$f(x_1) = \cos\frac{x_1}{2} = \cos\left(\frac{0.774596669}{2}\right) \text{ rad}$$

$$f(x_1) = \cos\frac{x_1}{2} = 0.9259328256$$

$$w_1\, f(x_1) = 0.555555 \times 0.925932825$$

$$w_1\, f(x_1) = 0.51440661 \qquad \ldots.(1)$$

$$f(x_2) = \cos\frac{x_2}{2} = \cos\left(\frac{0}{2}\right) \text{ rad}$$

$$f(x_2) = 1$$

$$w_2\, f(x_2) = 0.888888 \times 1$$

$$w_2\, f(x_2) = 0.888888 \qquad \text{....}(2)$$

$$f(x_3) = \cos\frac{x_3}{2} = \cos\left(\frac{-0.774596669}{2}\right) \text{ rad}$$

$$f(x_3) = 0.925932825$$

$$w_3\, f(x_3) = 0.555555 \times 0.925932825$$

$$w_3\, f(x_3) = 0.51440661 \qquad \text{....}(3)$$

Adding equations (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3) = 0.51440661 + 0.888888 + 0.51440661$$

$$= 1.91770$$

Result:

$$\int_{-1}^{1} \cos\frac{x}{2}\, dx = 1.91770$$

Example 4.22

Evaluate $\int_{-1}^{1} \cos\frac{x}{2} dx$ by applying 3 point gaussian quadrature and compare with exact solution.

Given:

Integral, $\qquad I = \int_{-1}^{1} \cos\frac{\pi x}{2}\, dx$

$$f(x) = \cos\frac{\pi x}{2}$$

To find: Evaluate the integral by using 3 Gaussian quadrature and compare with exact solution

Solution: We know that, for three point Gaussian quadrature

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \tfrac{5}{9} = 0.555555 \qquad \text{[Refer Table 4.1]}$$

We know that,

$$f(x) = \cos\frac{\pi x}{2}$$

$$f(x_1) = \cos\frac{\pi x_1}{2} = \cos\left(\frac{\pi \times (0.774596669)}{2}\right)$$

$$f(x_1) = 0.346711$$

$$w_1\, f(x_1) = 0.555555 \times 0.346711$$

$$w_1\, f(x_1) = 0.1926174 \qquad \ldots.(1)$$

$$f(x_2) = \cos\frac{\pi x_2}{2} = \cos\left(\frac{\pi \times 0}{2}\right)\text{rad}$$

$$f(x_2) = 1$$

$$w_2\, f(x_2) = 0.888888 \times 1 \qquad \ldots.(2)$$

$$f(x_3) = \cos\frac{\pi x_3}{2} = \cos\left(\frac{\pi \times (-0.774596669)}{2}\right)$$

$$f(x_3) = 0.346711$$

$$w_3\, f(x_3) = 0.555555 \times 0.346711$$

$$w_3\, f(x_3) = 0.1926174 \qquad \ldots.(3)$$

Adding equations (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3) = 0.1926174 + 0.888888 + 0.1926174$$

$$= 1.2741228$$

$$\int_{-1}^{1} \cos\frac{\pi x}{2}\,dx = 1.2741228$$

Exact solution:

$$\int_{-1}^{1} \cos\frac{\pi x}{2} = \left[\frac{\sin\frac{\pi x}{2}}{\frac{\pi}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{\pi} \times \left[\sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right)\right] = 1.2732395$$

Result:

1. $\int_{-1}^{1} \cos\frac{\pi x}{2} = 1.2741228$ [By three points gauss quadrature]
2. $\int_{-1}^{1} \cos\frac{\pi x}{2} = 1.2732395$ [By exact method]

Example 4.23

Evaluate $\int_{-1}^{1} \left[x^2 + \cos\frac{x}{2}\right] dx$ by applying 3 point gaussian quadrature and compare with exact solution.

Given:

Integral, $$I = \int_{-1}^{1} x^2 + \cos\left(\frac{x}{2}\right) dx$$

$$f(x) = x^2 + \cos\left(\frac{x}{2}\right) dx$$

To find: Evaluate the integral by using 3 Gaussian quadrature and compare with exact solution

Solution: We know that, for three point Gaussian quadrature

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \frac{5}{9} = 0.555555 \qquad \text{[Refer Table 4.1]}$$

We know that,

$$f(x) = x^2$$

$$f(x) = x^2 + \cos\left(\frac{x}{2}\right)$$

$$f(x_1) = {x_1}^2 + \cos\left(\frac{x_1}{2}\right)$$

$$= (0.774596669)^2 + \cos\left(\frac{0.774596669}{2}\right) \text{ rad}$$

$$f(x_1) = 1.5259328$$

$$w_1\, f(x_1) = 0.555555 \times 1.5259328$$

$$w_1\, f(x_1) = 0.8477396 \qquad \text{....(1)}$$

$$f(x_2) = {x_2}^2 + \cos\left(\frac{x_2}{2}\right) = (0)2 + \cos\left(\frac{0}{2}\right) \text{ rad}$$

$$f(x_2) = 1$$

$$w_2\, f(x_2) = 0.888888 \times 1$$

$$w_2\, f(x_2) = 0.888888 \qquad \text{.... (2)}$$

$$f(x_3) = {x_3}^2 + \cos\left(\frac{x_3}{2}\right)$$

$$= (-0.774596669)^2 + \cos\left(\frac{-0.774596669}{2}\right) \text{rad}$$

$$f(x_3) = 1.5259328$$

$$w_3\, f(x_3) = 0.555555 \times 1.5259328$$

$$w_3\, f(x_3) = 0.8477396 \qquad \text{.... (3)}$$

Adding equations (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3) = 0.8477396 + 0.888888 + 0.8477396$$

$$\int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right)\right] dx = 2.58436$$

Exact solution:

$$\int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right)\right] dx = \left[\frac{x^3}{3}\right]_{-1}^{+1} + \left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_{-1}^{+1}$$

$$= \frac{1}{3}[1^3 - (-1)^3] + 2\left[\sin\left(\frac{1}{2}\right) - \sin\left(\frac{-1}{2}\right)\right] \text{ rad}$$

$$= 2.58436$$

Result:

1. $\int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right)\right] dx = 2.58436$ [By three points gauss quadrature]

2. $\int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right)\right] dx = 2.58436$ [By exact method]

Example 4.24

Evaluate $\int_{-1}^{1} \frac{\cos x}{1-x^2} dx$ by applying 3 point gaussian quadrature.

Given:

Integral, $$I = \int_{-1}^{1} \frac{\cos x}{1 - x^2} dx$$

$$f(x) = \frac{\cos x}{1 - x^2} dx$$

To find: Evaluate the integral by using 3 Gaussian quadrature.

Solution: We know that, for three point Gaussian quadrature.

$$x_1 = \sqrt{\frac{3}{5}} = 0.774596669$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}} = -0.774596669$$

$$w_1 = \frac{5}{9} = 0.555555$$

$$w_2 = \frac{8}{9} = 0.888888$$

$$w_3 = \frac{5}{9} = 0.555555$$ [Refer Table 4.1]

We know that,

$$f(x) = \frac{\cos x}{1 - x^2}$$

$$f(x_1) = \frac{\cos x_1}{1 - x_1{}^2} = \frac{\cos(0.774596669)}{1 - (0.774596669)^2} \text{ rad}$$

$$f(x_1) = 1.78675798$$

$$w_1 f(x_1) = 0.555555 \times 1.78675798$$

$$w_1\, f(x_1) = 0.99264233 \qquad \text{....}(1)$$

$$f(x_2) = \frac{\cos x_2}{1 - {x_2}^2} = \frac{\cos(0)}{1 - (0)^2}$$

$$f(x_2) = 1$$

$$w_2\, f(x_2) = 0.888888 \times 1$$

$$w_2\, f(x_2) = 0.888888 \qquad \text{....}(2)$$

$$f(x_3) = \frac{\cos x_3}{1 - {x_3}^2} = \frac{\cos(-0.774596669)}{1 - (-0.774596669)^2} \text{ rad}$$

$$f(x_3) = 1.78675798$$

$$w_3\, f(x_3) = 0.555555 \times 1.78675798$$

$$w_3\, f(x_3) = 0.99264233 \qquad \text{....}(3)$$

Adding equations (1),(2) and (3),

$$w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3) = 0.99264233 + 0.888888 + 0.99264233$$

$$= 2.87417$$

Result:

$$\int_{-1}^{1} \frac{\cos x}{1 - x^2} dx = 2.87417$$

Example 4.25

Evaluate $\mathrm{I} = \int_{-1}^{1} \left[3\mathrm{e}^{\mathrm{x}} + \mathrm{x}^2 + \frac{1}{\mathrm{x}+2}\right] \mathrm{dx}$ using one point and two point gaussian quadrature and compare with exact solution.

Given:

Integral, $$\mathrm{I} = \int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x+2}\right] dx$$

$$f(x) = \left[3e^x + x^2 + \frac{1}{x+2}\right]$$

To find: 1. Evaluate the integral by using one point and two point gaussian quadrature.

2. Compare with exact solution.

Solution:

One point gaussian quadrature: We know that, for one point Gaussian quadrature.

$$x_1 = 0\,;\; w_1 = 2$$

$$f(x) = 3e^x + x^2 + \frac{1}{x+2}$$

$$f(x_1) = 3e^{x_1} + {x_1}^2 + \frac{1}{x_1+2}$$

$$= 3e^0 + 0^2 + \frac{1}{0+2}$$

$$f(x_1) = 3.5$$

$$w_1\, f(x_1) = 2 \times 3.5$$

$$w_1\, f(x_1) = 7$$

$$\int_{-1}^{1}\left[3e^x + x^2 + \frac{1}{x+2}\right]dx = \mathbf{7} \text{ for one point Gauss quadrature}$$

Two point gaussian quadrature: We know that, for two point Gaussian quadrature.

$$x_1 = +\sqrt{\frac{1}{3}} = 0.577350269$$

$$x_2 = -\sqrt{\frac{1}{3}} = -0.577350269$$

$$w_1 = 1$$

$$w_2 = 1$$ [Refer Table 4.1]

$$f(x) = 3e^x + x^2 + \frac{1}{x+2}$$

$$f(x_1) = 3e^{x_1} + {x_1}^2 + \frac{1}{x_1+2}$$

$$= 3e^{0.577350269} + (0.577350269)^2 + \frac{1}{0.577350269 + 2}$$

$$f(x_1) = 6.065265$$

$$w_1\, f(x_1) = 1 \times 6.065265$$

$$w_1\, f(x_1) = 6.065265 \qquad \text{....}(1)$$

$$f(x_2) = 3e^{x_2} + x_2{}^2 + \frac{1}{x_2 + 2}$$

$$= 3e^{-0.577350269} + (-0.577350269)^2 + \frac{1}{-0.577350269 + 2}$$

$$f(x_2) = 2.7203987$$

$$w_2\, f(x_2) = 1 \times 2.7203987$$

$$w_2\, f(x_2) = 2.7203987 \qquad \text{....}(2)$$

Adding (1) and (2),

$$w_1\, f(x_1) + w_2\, f(x_2) = 6.065265 + 2.7203987 = 8.7859$$

$$\int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x + 2}\right] dx = 8.7859 \text{ for two point Gauss quadrature}$$

Exact solution:

$$\mathrm{I} = \int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x + 2}\right] dx$$

$$= 3[e^x]_{-1}^{+1} + \left[\frac{x^3}{3}\right]_{-1}^{+1} + [\ln(x + 2)]_{-1}^{+1}$$

$$= 3\,[e^{+1} - (e^{-1})] + \frac{1}{3}[1^3 - (-1)^3][\ln(1 + 2\,) - \ln(-1 + 2)]$$

$$= 3\,[2.718 - 0.3678] + \frac{1}{3}\,[1 + 1] + \ln(3) - \ln(1)$$

$$\int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x+2}\right] dx = 8.8158$$

Result:

1. **One point gaussian quadrature**

$$\int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x+2}\right] dx = \mathbf{7}$$

2. **Two point gaussian quadrature**

$$\int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x+2}\right] dx = 8.7859$$

3. **Exact solution:**

$$\int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x+2}\right] dx = 8.8158$$

Example 4.25

Evaluate I = $\int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)$ dx dy using Gauss integration.

Given:

Integral, $$I = \int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)\ dx\,dy$$

$$f(x,y) = [2x^2 + 3\,xy + 4y^2]$$

To find: Evaluate the integral by using Gaussian quadrature.

Solution: We know that, the given integral is a polynomial of order 2. So, for exact integration,

So, $2n - 1 = 2$

$2n = 3$

$n = 1.5 \approx 2$

We should use two sampling points.

For two point Gaussian quadrature,

$$x_1 = 0.57735, \qquad y_1 = 0.57735$$

$$x_2 = -0.57735, \qquad y_2 = -0.57735$$

$$w_1 = 1$$

$$w_2 = 1$$ [Refer Table 4.1]

For two points scheme, the above equation can be written as,

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dx\,dy$$

$$= w_1{}^2 f(x_1,y_1) + w_1 w_2 f(x_1,y_1) + w_2 w_1 f(x_2,y_2) + w_2{}^2 f(x_2,y_2) \qquad \text{... (1)}$$

We know that,

$$f(x,y) = [2x^2 + 3\,xy + 4y^2]$$

$$w_1{}^2 f(x_1,y_1) = w_1{}^2(2x_1{}^2 + 3\,x_1 y_1 + 4y_1{}^2)$$

$$= 1^2\,[2\,(0.57735)^2 + 3\,(0.57735)(0.57735) + 4\,(0.57735)^2]$$

$$w_1{}^2 f(x_1,y_1) = 3 \qquad \text{... (2)}$$

$$w_1 w_2 f(x_1,y_1) = w_1 w_2(2x_1{}^2 + 3\,x_1 y_2 + 4y_2{}^2)$$

$$= 1\times1\,[2\,(0.57735)^2 + 3\,(0.57735)(-0.57735) + 4\,(-0.57735)^2]$$

$$w_1 w_2 f(x_1,y_1) = 1 \qquad \text{...(3)}$$

$$w_2 w_1 f(x_2,y_1) = w_2 w_1(2x_2{}^2 + 3\,x_2 y_1 + 4y_1{}^2)$$

$$= 1\times1\,[2\,(-0.57735)^2 + 3\,(-0.57735)(0.57735) + 4\,(0.57735)^2]$$

$$w_2 w_1 f(x_2,y_1) = 1 \qquad \text{... (4)}$$

$$w_2{}^2 f(x_2,y_2) = w_2{}^2(2x_2{}^2 + 3\,x_2 y_2 + 4y_2{}^2)$$

$$= 1^2\,[2\,(-0.57735)^2 + 3\,(-0.57735)(-0.57735) + 4\,(-0.57735)^2]$$

$$w_2{}^2 f(x_2, y_2) = 3 \qquad \text{... (5)}$$

Substitute the equation (2),(3),(4) and (5) in equation (1),

$$\int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)dx\,dy = 3 + 1 + 1 + 3 = 8$$

$$\int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)dx\,dy = 8$$

Verification: The exact solution of integral is,

$$\int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)\,dx\,dy$$

$$= \int_{-1}^{1}\left\{\left[\frac{2}{3}x^3 + \frac{3}{2}yx^2 + 4y^2x\right]_{-1}^{1}\right\}dy$$

$$= \int_{-1}^{1}\left\{\frac{2}{3}(1+1) + \frac{3}{2}y(1-1) + 4y^2(1+1)\right\}dy$$

$$= \int_{-1}^{1}\left(\frac{4}{3} + 8y^2\right)dy$$

$$= \left[\frac{4}{3}y + \frac{8}{3}y^3\right]_{-1}^{1} = \frac{4}{3}(1+1) + \frac{8}{3}(1+1)$$

$$= \frac{8}{3} + \frac{16}{3} = \frac{24}{3} = 8$$

$$\int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)\,dx\,dy = 8$$

Result:

The Integral $\int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3\,xy + 4y^2)\,dx\,dy = 8$

Example 4.27

Integrate the function $f(r) = 1+r+r^2+r^3$ between the limits -1 and +1 using

i) Exact Method

ii) Gauss integration method and compare the two results

Given function , $f(r) = 1+r+r^2+r^3$

To find: Evalute the integral by using gauss integration method and compare with exact method.

Solution:

We know that, the given integrated is a polynomial of order 3 .

So, $2n - 1 = 3$

$\Rightarrow \quad 2n = 4$

$\Rightarrow \quad n = 2$

We should use two sampling points.

For two point Gaussian quadratic,

$$r_1 = +\sqrt{\frac{1}{3}} = 0.577350269$$

$$r_2 = -\sqrt{\frac{1}{3}} = -0.577350269$$

$$w_1 = 1$$

$$w_2 = 1$$

We know that, $f(r) = 1+r+r^2+r^3$

$$f(r_1) = 1 + r_1 + r_1^2 + r_1^3$$

$$= 1 + 0.577350269 + (0.577350269)^2 + (0.577350269)^3$$

$$f(r_1) = 2.1031336$$

$$w_1 f(r_1) = 1 \times 2.1031336$$

$$w_1 f(r_1) = 2.1031336 \qquad \text{... (1)}$$

$$f(r_2) = 1 + r_2 + r_2^2 + r_2^3$$

$$= 1 + (0.577350269) + (-0.577350269)^2 + (0.577350269)^3$$

$$f(r_2) = 0.5635329$$

$$w_2 f(r_2) = 1 \times 0.5635329$$

$$w_2 f(r_2) = 0.5635329 \qquad \text{... (2)}$$

Adding (1) and (2) ,

$$w_1 f(r_1) + w_2 f(r_2) = 2.1031336 + 0.5635326 = 2.666666$$

$$\Rightarrow \int_{-1}^{1} (1 + r + r^2 + r^3) = 2.666666$$

Exact method:

$$\int_{-1}^{1} (1 + r + r^2 + r^3) = \left[r + \frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4} \right]_{-1}^{+1}$$

$$= [r]_{-1}^{+1} - \frac{1}{2}[r^2]_{-1}^{+1} + \frac{1}{3}[r^3]_{-1}^{+1} + \frac{1}{4}[r^4]_{-1}^{+1}$$

$$= [1 - (-1)] + \frac{1}{2}[(1)^2 - (-1)^2] + \frac{1}{3}[1^3 - (-1)^3] + \frac{1}{4}[1^4 - (-1)^4]$$

$$= 2 + \frac{1}{2}(0) + \frac{1}{3}(1 + 1) + \frac{1}{4}(0)$$

$$\int_{-1}^{1} (1 + r + r^2 + r^3) = 2.666666$$

Result:

1. $$\int_{-1}^{1}(1+r+r^2+r^3)dr = 2.666666 \qquad \text{[By ginuss integration]}$$

2. $$\int_{-1}^{1}(1+r+r^2+r^3)dr = 2.666666 \qquad \text{[By exact integration]}$$

Example 4.28

Evaluate the integral $\mathrm{I} = \int_{-1}^{1}(\mathrm{a}_1 + \mathrm{a}_2\mathrm{x} + \mathrm{a}_3\mathrm{x}^2 + \mathrm{a}_4\mathrm{x}^3)\mathrm{dx}$ using Gauss integration

Given: Integral, $\mathrm{I} = \int_{-1}^{1}(\mathrm{a}_1 + \mathrm{a}_2\mathrm{x} + \mathrm{a}_3\mathrm{x}^2 + \mathrm{a}_4\mathrm{x}^3)\mathrm{dx}$

$$f(x) = a_1 + a_2x + a_3x62 + a_4x63$$

To find: evaluate the integral by using Gauss integration.

Solution: we know that, the given integrand is a polynomial of order 3. So, for exact integration,

2 n – 1 = 3

2 n = 4

n = 2

We should use two sampling points,

For two point Gaussian quadrature,

$$x_1 = +\sqrt{\frac{1}{3}} = 0.577350269$$

$$x_2 = -\sqrt{\frac{1}{3}} = -0.577350269$$

$$w_1 = 1$$

$$w_2 = 1$$

We know that $f(x) = a_1 + a_2x + a_3x^2 + a_4x^3$

$$\Rightarrow \qquad f(x_1) = a_1 + a_2x_1 + a_3x_1^2 + a_4x_1^3$$

$$= a_1 + a_2(0.577350269) + (0.33333333)a_3 + (0.192450089)a_4$$

$$\Rightarrow \quad w_1 f(x_1) = 1\times(a_1 + 0.577350269\, a_2 + 0.33333333\, a_3 + 0.192450089\, a_4) \qquad \text{... (1)}$$

$$f(x_2) = a_1 + a_2x_2 + a_3x_2^2 + a_4x_2^3$$

$$= a_1 + a_2(-0.577350269) + a_3\,(-0.577350269)^2 + a_4(-0.577350269)^3$$

$$f(x_2) = a_1 - \ (0.577350269\ )\, a_2 + 0.33333333\, a_3 - 0.192450089\, a_4$$

$$\Rightarrow \quad w_2 f(x_2) = 1\times[\, a_1 - \ 0.577350269\, a_2 + 0.33333333\, a_3 + 0.192450089\, a_4] \qquad \text{... (2)}$$

Adding equation (1) and (2),

$$w_1 f(x_1) + \ w_2 f(x_2) = 2a_1 + 0.666666\, a_3$$

$$\Rightarrow \int_{-1}^{1} (a_1 + a_2x + a_3x^2 + a_4x^3)dx = 2a_1 + 0.666666\, a_3$$

$$\Rightarrow \int_{-1}^{1} (a_1 + a_2x + a_3x^2 + a_4x^3)dx = 2\, a_1 + \frac{2}{3}\, a_3$$

Verification:

$$\int_{-1}^{1} (a_1 + a_2x + a_3x^2 + a_4x^3)dx = \left[a_1x + a_2\frac{x^2}{2} + a_3\frac{x^3}{3} + a_4\frac{x^4}{4}\right]_{-1}^{1}$$

$$= a_1[x]_{-1}^{1} + \frac{a_2}{2}\ [x62\ ]_{-1}^{1} + \frac{a_3}{3}\ [x^3]_{-1}^{1} + \frac{a_4}{4}\,[x^4]_{-1}^{1}$$

$$= a_1[1-(-1)] + \frac{a_2}{2}[1-(-1)^2] + \frac{a_3}{3}[1-(-1)^3] + \frac{a_4}{4}\ [1-(-1)^4]$$

$$= 2a_1 + \frac{2}{3}a_3$$

Result: $\int_{-1}^{1}(a_1 + a_2x + a_3x^2 + a_4x^3)dx = 2\ a_1 + \frac{2\ a_3}{3}$

UNIT 5

APPLICATION TO HEAT TRANSFER AND DYNAMIC ANALYSIS

5.1 HEAT TRANSFER

Heat transfer can be defined as the transmission of energy from one region to another region due to temperature difference. A knowledge of the temperature distribution within a body is important in many engineering problems. There are three modes of heat transfer.

They are:

(i) Conduction

(ii) Convection

(iii) Radiation

(i) Conduction

Heat conduction is a mechanism of heat transfer from a region of high temperature to a region of low temperature within a medium (solid, liquid or gases) or between different medium in direct physical contact.

In conduction, energy exchange takes place by the kinematic motion or direct impact of molecules. Pure conduction is found only in solids.

(ii) Convection

Convection is a process of heat transfer that will occur between a solid surface and a fluid medium when they are at different temperatures.

Convection is possible only in the presence of fluid medium.

(iii) Radiation

The heat transfer from one body to another without any transmitting medium is known as radiation. It is an electromagnetic wave phenomenon.

5.2 DERIVATION OF TEMPERATURE FUNCTION (T) AND SHAPE FUNCTION (N) FOR ONE DIMENSIONAL HEAT CONDUCTION ELEMENT

Consider a bar element with nodes 1 and 2 as shown in Fig.5.1. T_1, and T_2 are the temperatures at the respective nodes. So, T_1, and T_2 are considered as degrees of freedom of this bar element.

Since the element has got two degrees of freedom, it will have two generalized co-ordinates.

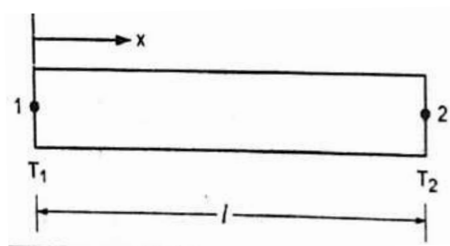


Fig.5.1

Writing the equation in matrix form,

$$T = [\,1\ x\,]\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

At node 1, T = T_1, $x = 0$

At node 1, T = T_2, $x = 1$

Substitute the above values in equation

$\Rightarrow$ $$T_1 = a_0$$

$\Rightarrow$ $$T_2 = a_0 + a_1 l$$

Assembling the equations in matrix form

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l-0} \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\left[Note: \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \times \begin{bmatrix} a_{22} & a -_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$\Rightarrow \qquad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & l \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} \mathrm{a}_0 \\ \mathrm{a}_1 \end{Bmatrix}$ values in equation

$$\Rightarrow \qquad T = [\,1 \;\; x] \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l} \; [\,1 \;\; x] \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l} \; [\, l - x \;\; 0 + x] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$[\because matrix\ multiplication\ (1 \times 2)(2 \times 2) = (1 \times 2)]$$

$$T = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$T = [N_1 \quad N_2] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Temperature function, $\quad T = N_1 + T_1 + N_2 T_2$

where, shape functions, $\quad N_1 = \dfrac{l-x}{l}$

$$N_2 = \frac{x}{l}$$

5.3 DERIVATION OF STIFFNESS MATRIX FOR ONE DIMENSIONAL HEAT CONDUCTION ELEMENT

We know that,

Consider a one dimensional bar element with nodes 1 and 2 as shown in Fig.5.2. Let T and T_2 be the temperatures at the respective nodes and k be the thermal conductivity of the material.

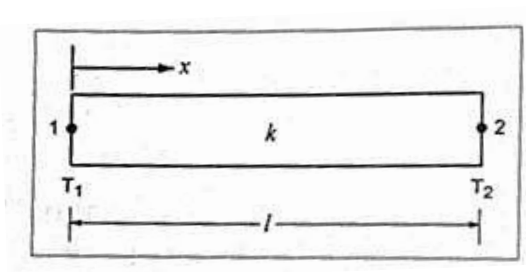


Fig. 5.2

We know that

Stiffness matrix $$[K] = \int_v [B]^T[D][B]\ dv$$

In one dimensional element,

Temperature function, $$T = N_1T_1 + N_2T_2$$

where, shape functions, $$N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

We know that,

strain – Displacement matrix, $$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$\Rightarrow \qquad [B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$\Rightarrow \qquad [B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

In one dimensional heat conduction problems,

[D] = [K] =k = Thermal conductivity of the material

Substitute [B], $[B]^T$ and [D] values in stiffness matrix equation

$$\Rightarrow \left.\begin{matrix}\text{Stiffness matrix}\\ \text{for heat conduction}\end{matrix}\right\}[K_C] = \int_0^1 \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times k \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$= \int_0^1 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} k \ dv$$

$$[\because \mathit{matrix\ multiplication}\ (2 \times 1)(1 \times 2) = (2 \times 2)]$$

$$= \int_0^1 \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} k\ A\ dx \qquad [\because dv = A \times dx]$$

$$= A\ k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^1 d\mathrm{x}$$

$$= A\ k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [\ x]_0^l$$

$$= A\ k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l - 0)$$

$$= A\ k\ l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$= \frac{A\,k\,l}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\,K_C] = \frac{A\,k\,l}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Where, A = Area of the element, m^2

K = Thermal conductivity of the element, W/mK

l = Length of the element, m

5.4 FINITE ELEMENT EQUATION OF ONE DIMENSIONAL HEAT CONDUCTION PROBLEMS

We know that,

General force equation is $\{F\} = [\,K_c]\{\,T\}$

Where, {F} is a element force vector [Column matrix]

$[K_C]$ is a stiffness matrix [Row matrix]

{T} is a model temperature [Column matrix]

For one dimensional heat conduction problems, stiffness matrix, [K] is given by

$$[K_C] = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Consider a two noded element as shown in Fig. 5.4

Force vector $\{\,F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

Nodal temperature $\{\,T\} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$

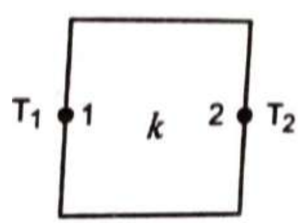


Fig .5.3

Subsitute $[K_C]$ { F} and{ T} values in equaiton

$$\Rightarrow \qquad \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Case (i) One dimensional heat conduction with free end convection

Consider a one dimensional element with nodes 1 and 2 as shown in T_1 and T_2 are the temperatures at the respective nodes. Assume convection occurs only from the right end of the element as shown in Fig.5.4

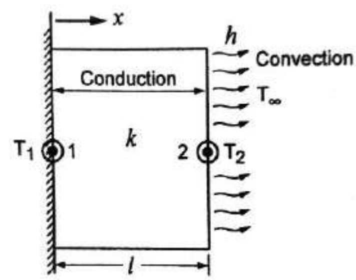


Fig.5.4

Stiffness matrix $[K_C]$ for one dimensional heat conduction element is given by

$$[K_C] = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The convection term contribution to the stiffness matrix is given by

$$[\,K_h]_{end} = \iint_A h\,[N]^T[N]dA$$

Where h = Heat transfer coefficient, W/m^2k

N = Shape factor

We know that,

Shape factor, $$[\,N] = [\,N_1\;N_2] = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

At node 2, $x = l$

$$\Rightarrow \qquad [N] = [N_1\;N_2] = [\,0\;1]$$

$$\Rightarrow \qquad [N]^T = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Substitute [N] and $[\,N\,]^T$ values in equation,

$$\Rightarrow \qquad [K_h]_{end} = \iint_A h\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}[\,0\;1\,]\,dA$$

$$[K_h]_{end} = h\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\int dA$$

$$(2 \times 1) \times (1 \times 2) = (2 \times 2)$$

$$[K_h]_{end} = h\,A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Stiffness matrix $\quad [\,K] = [\,K_C] + [K_h]$

$$[K] = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h\,A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The convection force form the free end of the element is obtained from the following relation,

$$\{F_h\}_{end} = h\,T_\infty A \begin{Bmatrix} N_1(x = l) \\ N_2\ (x = l) \end{Bmatrix}$$

$$\{F_h\}_{end} = h\,T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

We know that, general force equation is

$$\{F\} = [K]\,\{T\}$$

Substitute $\{F\}$ *and* $[K]$ values,

$$\Rightarrow \qquad h\,T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h\,A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\Rightarrow \qquad \left[\frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h\,A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \ h\,T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\right]$$

Where, A = Area of the element, m^2

k = Thermal conductivity of the element, W/mK

l = Length of the element

h = Heat transfer coefficient, W/m^2K

T_∞ = Fluid temperature, K

T = Temperature, K

This is a finite element equation for one dimensional heat conduction element with free end convection.

Case (ii) One dimensional element with conduction, convection and internal heat conduction, convection and internal heat generation.

Consider a rod with nodes 1 and 2 as shown in Fig.5.5. This rod is subjected to conduction, convection and internal heat generation.

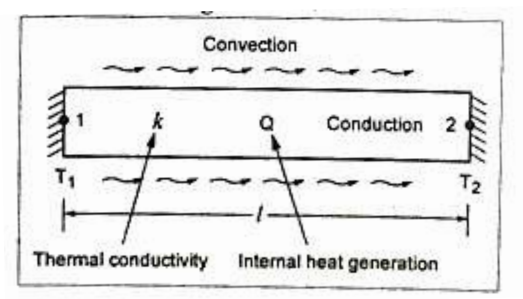


Fig.5.5.

We know that, heat conduction part of the stiffness matrix [K] for the one dimensional element is

$$[K_c] = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Heat convection part of the stiffness matrix [K] for the one dimensional element is given by

$$[\,K_h] = \iint_S h\,[N]^T[N]\,dS$$

$$= h\,P\int_0^l [\,N]^T[\,N]\,dx$$

$$[\because dS = P \times dx \; where \; P = Perimeter \; of \; the \; element]$$

$$= h\,P\int_0^l \begin{Bmatrix} \dfrac{l-x}{l} \\ \dfrac{x}{l} \end{Bmatrix}\begin{bmatrix} \dfrac{l-x}{l} & \dfrac{x}{l} \end{bmatrix} dx$$

$$= h\,P\int_0^l \begin{bmatrix} \left(1-\dfrac{x}{l}\right)^2 & \dfrac{x}{l}-\dfrac{x^2}{l^2} \\ \dfrac{x}{l}-\dfrac{x^2}{l^2} & \dfrac{x^2}{l^2} \end{bmatrix} dx$$

$$= h\,P \begin{bmatrix} \dfrac{\left(1-\dfrac{x}{l}\right)^3}{3\times\left(-\dfrac{1}{l}\right)} & \dfrac{x^2}{2\,l}-\dfrac{x^3}{3\,l^2} \\ \dfrac{x^2}{2\,l}-\dfrac{x^3}{3\,l^2} & \dfrac{x^3}{3\,l^2} \end{bmatrix}_0^1$$

$$= h\,P \begin{bmatrix} \dfrac{\left(1-\dfrac{l}{l}\right)^3}{\dfrac{-3}{l}}-\dfrac{l^3}{\dfrac{-3}{l}} & \dfrac{l^2}{2\,l}-\dfrac{l^3}{3\,l^2}-0 \\ \dfrac{l^2}{2\,l}-\dfrac{l^3}{3\,l^2}-0 & \dfrac{l^3}{3\,l^2}-0 \end{bmatrix}$$

$$= h\,P \begin{bmatrix} \dfrac{l}{3} & \dfrac{l}{2}-\dfrac{l}{3} \\ \dfrac{l}{2}-\dfrac{l}{3} & \dfrac{l}{3} \end{bmatrix}$$

$$= h\,P \begin{bmatrix} \dfrac{l}{3} & \dfrac{l}{6} \\ \dfrac{l}{6} & \dfrac{l}{3} \end{bmatrix}$$

$$[\,K_h] = \frac{h\,P\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Stiffness matrix $[\,K\,] = [\,K_C] + [K_h]$

$$[K] = \frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Force matrix due to heat generation is given by,

$$\{F_Q\} = \iiint_v [\,N]^T\,Q\,dV$$

$$= \int_0^l [N]^T \times Q \times A \times dx \qquad [\because dV = A \times dx]$$

$$= Q \times A \int_0^l [N]^T dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} \frac{l - x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} l - \frac{x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} x - \frac{x^2}{2\,l} \\ \frac{x^2}{2\,l} \end{Bmatrix}_0^l$$

$$= Q \times A \begin{Bmatrix} l - \frac{l^2}{2\,l} - 0 \\ \frac{l^2}{2\,l} - 0 \end{Bmatrix}$$

$$= Q \times A \begin{Bmatrix} \frac{l^2}{2\,l} \\ \frac{l^2}{2\,l} \end{Bmatrix} = Q \times A \begin{Bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

$$\{ F_Q \} = Q \times A \times \frac{l}{2} \begin{Bmatrix} l \\ l \end{Bmatrix}$$

Force matrix due to convection is given by

$$\{F_h\} = \iint_S h\, T_\infty\, [\, N]^T\, dS$$

$$= \iint_S h\, T_\infty\, [\, N]^T\ P\ \times dx \qquad [\because dS = P \times dx]$$

$$= P\ h\, T_\infty \int_0^l [N]^T dx$$

$$= P\ h\ T_\infty \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= P\ h\ T_\infty \begin{Bmatrix} x - \frac{x^2}{2\,l} \\ \frac{x^2}{2\,l} \end{Bmatrix}_0^l$$

$$= P\ h\ T_\infty \begin{Bmatrix} l - \frac{l^2}{2\,l} - 0 \\ \frac{l^2}{2\,l} - 0 \end{Bmatrix}$$

$$= P\ h\ T_\infty \begin{Bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

$$\{F_h\} = \frac{P\ h\ T_\infty 1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Adding equations

Force matrix, $\{F\} = \{F_Q\} + \{F_h\}$

$$= \frac{Q\ A\ l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{P\ h\ T_\infty 1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{F\} = \frac{Q\ A\ l + P\ h\ T_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

We know that, General force equaiton is

$$\{F\} = [\,K]\{\,T\}$$

Substitute { F } and [K] values,

$$\Rightarrow \qquad \frac{Q\ A\ l + P\ h\ T_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left[\frac{A\ k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\ P\ l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\Rightarrow \quad \left[\frac{A\,k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Q\,A\,l + P\,h\,T_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Where, A = Area of the element, m^2

k = Thermal conductivity of the element, W/mk

l = Length of the element, m

h = Heat transfer coefficient, W/m^2k

P = Perimeter, m

T = Temperature, K

Q = Heat generation, W

$T\infty$ = Fluid temperature, K

This is a finite element equation for one dimensional element which is subjected to conduction, convection and internal heat generation.

5.5 SOLVED PROBLEMS - HEAT TRANSFER [ONE DIMENSIONAL]

Example 5.1

A wall of 0.6 m thickness having thermal conductivity of 1.2 W/mK. The wall is to be insulated with a material of thickness 0.06 m having an average thermal conductivity of 0.3 W/mK. The inner surface temperature is 1000 °C and outside of the insulation is exposed to atmospheric air at 30°C with heat transfer coefficient of $35 W/m^2K$. Calculate the nodal temperatures.

Given:

Thickness of the wall, $l_1 = 0.6$ m

Thermal conductivity of the wall, $k_1 = 1.2$ W/Mk

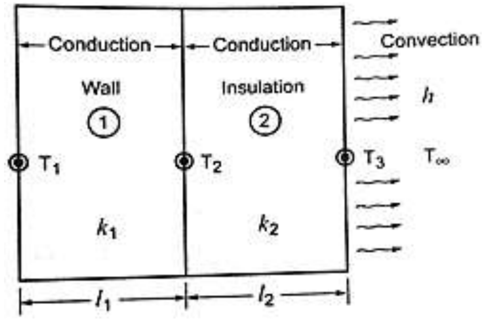
Thickness of the insulation, $l_2 = 0.06$ m

Thermal conductivity of the insulation, $k_2 = 03$ W/mK

Inner surface temperature, $T_1 = 1000^0C + 273 = 1273$ K.

Atmospheric air temperature, $T_\infty = 30^0C + 273 = 303\ K$

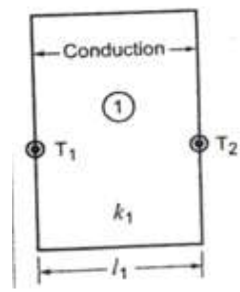
Heat transfer coefficient at outer side, h = 35 W/m^{2}k



To find: Nodal temperatures, (T_2 and T_3)

Solution:

For element 1: (Nodes 1, 2)



Finite element equation is

$$\frac{A_1 k_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

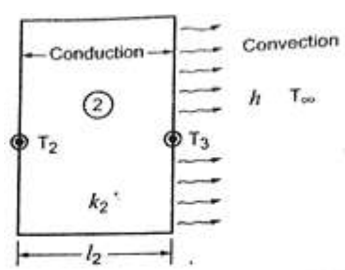
For unit area,

$$A_1 = 1\, m^2$$

$$\frac{1.2}{0.6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{array}{cc} 1 & 2 \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] & \begin{array}{c} 1 \\ 2 \end{array} \end{array}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \ldots (1)$$

For element 2: (Nodes 2, 3)



This element is subjected to both conduction and convection. So, finite element equation is

$$\left(\frac{A_2\, k_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h\, A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = h\, T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\frac{1 \times 0.3}{0.06}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 35 \ \times 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = 35 \ \times 303 \times 1 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}\right)\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

$$\begin{array}{cc} \mathbf{2} & \mathbf{3} \\ \end{array}$$

$$\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.605 \times 10^3 \end{Bmatrix} \qquad \text{....(2)}$$

Assemble the finite elements, i.e., assemble the finite element equations (1) and (2).

$$\underset{[\,K\,]}{\underset{\downarrow}{\begin{bmatrix} 2 & -2 & 0 \\ -2 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix}}} \underset{[\,T\,]}{\underset{\downarrow}{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}} = \underset{[\,F\,]}{\underset{\downarrow}{\begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}}} \qquad \text{...(3)}$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 2: the first row of the force matrix is replaced by the known temperature at node 1, i.e., T_1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value (From equation no.3) is multiplied by known temperature at node 1, i.e., -2 × 1273 = - 2546. This value (as positive digit, i.e., 2546) has been added to the second row of the force matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 2546 \\ 10.605 \times 10^3 \end{Bmatrix} \quad \ldots (4)$$

Solving equation (4),

$$\Rightarrow \quad 7\,T_2 - 5\,T_3 = 2546 \quad ..(5)$$

$$-5\,T_2 + 40\,T_3 = 10.605 \times 10^3 \quad ..(6)$$

Equation (5) × 8,

$$\Rightarrow \quad 56\,T_2 - 40\,T_3 = 20.368 \times 10^3 \quad ..(7)$$

Equation (6) $\Rightarrow$ $$-\,5\,T_2 + 40\,T_3 = 10.605 \times 10^3$$

$$51\,T_2 = 30.973 \times 10^3$$

$$T_2 = 607.313\,K$$

Substitute T_2 value in equation (5),

$$\Rightarrow \quad 7 \times 607.313 - 5\,T_3 = 2546$$

$$T_3 = 341.03\,K$$

Result: Nodal temperatures: T_1 = 1273 K

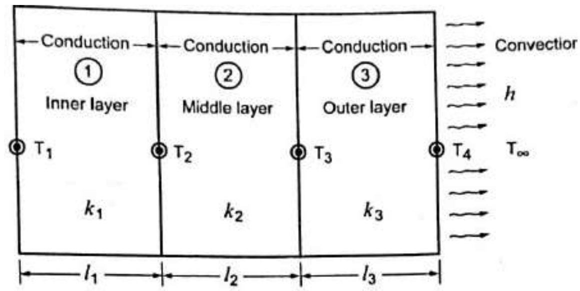
T_2 = 607.313 K

T_3 = 341.03 K

Example 5.2

A furnace wall is made up of three layers, inside layer with thermal conductivity 8.5 W/mK, the middle layer with conductivity 0.25 W/mK, the outer layer with conductivity 0.08 W/mK. The respective thickness of the inner, middle and outer layer are 25 cm, 5 cm and 3 cm respectively. The inside temperature of the wall is 600 °C and outside of the wall is exposed to atmospheric air at 30°C with heat transfer coefficient of 45 W/m²K. Determine the nodal temperatures.

Given:



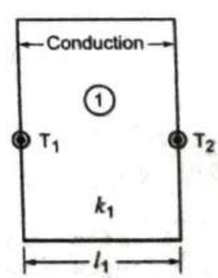
Thermal conductivity of the inner layer,	$k_1 = 8.5$ W/mk
Thermal conductivity of the middle layer,	$k_2 = 0.25$ W/mK
Thermal conductivity of the outer layer,	$k_3 = 0.08$ W/mK
Inner thickness,	$l_1 = 25$ cm $= 0.25$ m
Middle layer thickness,	$l_2 = 5$ cm $= 0.05$ m
Outer layer thickness,	$l_3 = 3$ cm $= 0.03$ m
Inner temperature of the wall,	$T_1 = 600^0$C$+ 273 = 873$ K.
Atmospheric air temperature,	$T_\infty = 30^0C + 273$ $= 303\ K$
Heat transfer coefficient at outer side,	$h = 45$ W/m^{2}k

To find: Nodal temperatures, (T_2 ,T_3 and T_4)

Solution:

For element 1: (Nodes 1, 2)

Finite element equation is



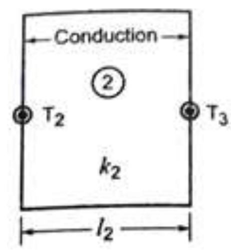
$$\frac{A_1 k_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

For unit area, $A_1 = 1\ m^2$

$$\Rightarrow \qquad \frac{8.5}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{array}{cc} \mathbf{1} & \mathbf{2} \\ \left[\begin{array}{cc} 34 & -34 \\ -34 & 34 \end{array}\right] \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \ldots (1)$$

For element 2: (Nodes 2, 3)



Finite element equation is

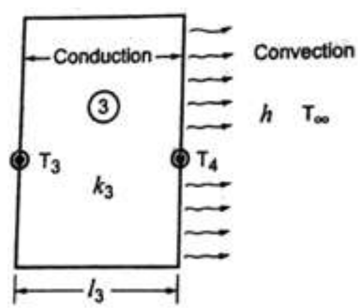
$$\frac{A_2\, k_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \quad \frac{1 \times 0.25}{0.05}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{array}{cc} 2 & 3 \\ \left[\begin{array}{cc} 5 & -5 \\ -5 & 5 \end{array}\right] \end{array} \begin{array}{c} 2 \\ 3 \end{array} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \ldots (2)$$

For element 2: (Nodes 2, 3)

This element is subjected a both conduction and convection. So, finite element equation is



$$\left(\frac{A_3\, k_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h\,A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = h\,T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\frac{1 \times 0.08}{0.03}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 45 \times 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = 45 \times 303 \times 1 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 2.666 & -2.666 \\ -2.666 & 2.666 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 45 \end{bmatrix}\right)\begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 13.635 \times 10^3 \end{Bmatrix}$$

$$\begin{array}{cc} 3 & 4 \\ \left[\begin{array}{cc} 2.666 & -2.666 \\ -2.666 & 2.666 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \end{array} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 13.635 \times 10^3 \end{Bmatrix} \qquad \ldots . (3)$$

Assemble the finite elements, i.e., assemble the finite element equations (1), (2) and (3).

$$\begin{array}{c c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} & \\ \begin{bmatrix} 34 & -34 & 0 & 0 \\ -34 & 34+5 & -5 & 0 \\ 0 & -5 & 5+2.666 & -2.666 \\ 0 & 0 & -2.666 & 47.666 \end{bmatrix} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \end{array} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

In this problem, there is no heat generation and there is no convection except from the right end.

So, $\{F_1\} = \{F_2\} = \{F_3\} = 0$

and $\{F_4\} = 13.635 \times 10^3$

$$\underbrace{\begin{bmatrix} 34 & -34 & 0 & 0 \\ -34 & 39 & -5 & 0 \\ 0 & -5 & 7.666 & -2.666 \\ 0 & 0 & -2.666 & 47.666 \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}}_{[T]} = \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13.635 \times 10^3 \end{Bmatrix}}_{[F]} \qquad \ldots (4)$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & -5 & 0 \\ 0 & -5 & 7.666 & -2.666 \\ 0 & 0 & -2.666 & 47.666 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13.635 \times 10^3 \end{Bmatrix}$$

Step 2: the first row of the force matrix is replaced by the known temperature at node 1, i.e., T_1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & -5 & 0 \\ 0 & -5 & 7.666 & -2.666 \\ 0 & 0 & -2.666 & 47.666 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 0 \\ 0 \\ 13.635 \times 10^3 \end{Bmatrix}$$

Equation (5) becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.128 & 0 \\ 0 & 0 & 1 & -0.376 \\ 0 & 0 & 0 & 46.655 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 761.076 \\ 541.614 \\ 13.635 \times 10^3 \end{Bmatrix}$$

$$\Rightarrow \qquad 46.655\, T_4 = 15.076 \times 10^3$$

$$\Rightarrow \qquad T_4 = 323.21\, K$$

$$\Rightarrow \qquad T_3 - 0.379\, T_4 = 541.614$$

$$T_3 - 0.379(323.21) = 541.614$$

$$T_3 = 664.11\, K$$

$$\Rightarrow \qquad T_2 - 0.218\, (T_3) = 761.076$$

$$T_3 - 0.128(664.11) = 761.076$$

$$T_2 = 846.08\, K$$

Result: Nodal temperatures:

$$T_1 = 873\, K$$

$$T_2 = 846.08\, K$$

$$T_3 = 664.11\, K$$

$$T_4 = 323.21\, K$$

Verification:

Heat flow through composite wall is given by

$$Q = \frac{\Delta\, T_{overall}}{R}$$

[From HMT data book, C. P. Kothandaraman, Page No. 43 & 44]

Where, $\Delta\, T = T_1 - T_\infty$

$$R = \frac{1}{\frac{1}{h_{in} A} + \frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} + \frac{l_3}{k_3 A} + \frac{I}{h_{out}\, A}}$$

$$\Rightarrow \qquad Q = \frac{T_1 - T_\infty}{\frac{1}{h_{in}A} + \frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} + \frac{l_3}{k_3 A} + \frac{I}{h_{out}\, A}}$$

Heat transfer coefficient at inner side is not given. So, neglect that term.

$$\Rightarrow \qquad Q = \frac{873 - 303}{\frac{0.25}{8.5} + \frac{0.05}{0.25} + \frac{0.03}{0.08} + \frac{1}{45}} \qquad [\because A = 1m^2]$$

$$\Rightarrow \qquad Q = 909.62\ W/m^2$$

We know that, interface temperature

$$Q = \frac{T_1 - T_\infty}{R} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_1}{R_2} = \frac{T_3 - T_4}{R_3} = \frac{T_4 - T_\infty}{R_{outer}} \qquad \text{... (6)}$$

$$\Rightarrow \qquad Q = \frac{T_1 - T_2}{R_1}$$

$$\Rightarrow \qquad Q = \frac{T_1 - T_2}{\frac{l_1}{k_1 A}} \qquad [\because R_1 = \frac{l_1}{k_1 A}]$$

$$\Rightarrow 909.62 = \frac{873 - T_2}{\frac{0.25}{8.5}} \qquad [\because A = 1\ m^2]$$

$$\Rightarrow \qquad T_2 = 846.24\ K$$

$$(5) \Rightarrow \qquad Q = \frac{T_2 - T_3}{R_2}$$

$$= \frac{T_2 - T_3}{\frac{l_2}{k_2 A}} \qquad [\because R_2 = \frac{l_2}{k_2 A}]$$

$$909.62 = \frac{846.24 - T_3}{\frac{0.05}{0.25}} \quad [\because A = 1\ m^2]$$

$$\Rightarrow \qquad T_3 = 664.31\ K$$

$$(6) \Rightarrow \qquad Q = \frac{T_3 - T_4}{R_3}$$

$$= \frac{T_3 - T_4}{\frac{l_3}{k_3 A}} \quad [\because R_3 = \frac{l_3}{k_3 A}]$$

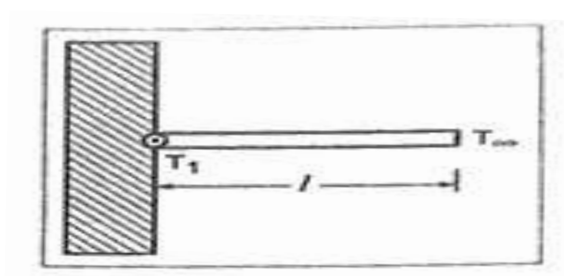
$$909.62 = \frac{664.31 - T_4}{\frac{0.03}{0.08}} \; [\because A = 1\, m^2]$$

$$\Rightarrow \qquad T_4 = 323.20\, K$$

Example 5.3

An aluminium alloy fin of 7 mm thick and 50 mm long protrudes from a wall, which is maintained at 120°C. The ambient air temperature is 22 °C. The heat transfer coefficient and thermal conductivity of the fin material are 140 W/m²K and 55 W/mK respectively. Determine the temperature distribution of fin.

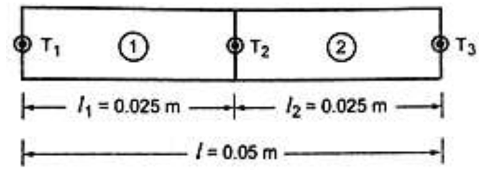
Given:



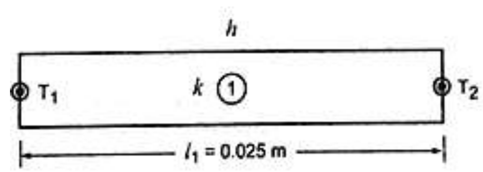
Thickness,	t = 7 mm = 0.007 m
Length,	l = 50 mm = 0.050 m
Base temperature,	$T_1 = 120^0C + 273 = 393$ K
Ambient temperature,	$T = 22^0C + 273 = 295$ K
Heat transfer coefficient,	$h = 140$ W/m^2 K
Thermal conductivity,	K = 55 W/mK

To find: Temperature distribution, $(T_1, T_2, T_3, T_4, \ldots\ldots) in\ the\ fin.$

Solution: For simplicity's sake, discretize the fin into two equal size element.



For element 1: (Nodes 1, 2)



We know that,

Finite element equation is

$$\frac{A}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Q\,A\,l_1 + P\,h\,T_\infty\,l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{...(1)}$$

where $\quad P = \text{Perimeter} = 2 \times l \text{ (Approximately)}$

$$= 2 \times 0.050$$

$$A = \text{Area} = \text{Length} \times \text{Thickness}$$

$$= 0.050 \times 0.007$$

$$A = 3.5 \times 10^{-4} m^2$$

Substitute A, P, k, h, l_l and T_∞ values in equation (1).

$$(1) \Rightarrow \quad \frac{3.5 \times 10^{-4} \times 55}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{140 \times 0.1 \times 0.025}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{Q\,A\,l_1 + 140 \times 0.1 \times 295 \times 0.025 P\,h\,T_\infty\,l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

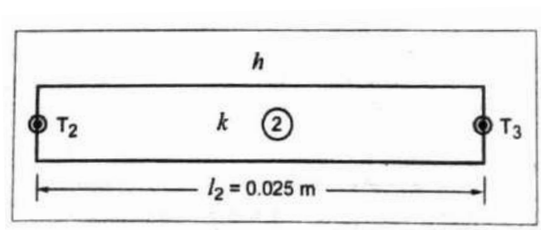
Heat generation Q is not given. So, neglect that term, $\left(\frac{Q\,A\,l_1}{2}\right)$

$$\Rightarrow \quad \left(0.77\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 0.0583\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = 51.625\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix}0.77 & -0.77\\ -0.77 & 0.77\end{bmatrix} + \begin{bmatrix}0.116 & 0.0583\\ 0.0583 & 0.116\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}51.625\\ 51.625\end{Bmatrix}$$

$$\Rightarrow \quad \begin{array}{c c} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix}0.886 & -0.7117\\ -0.7117 & 0.886\end{bmatrix} & \begin{array}{c}1\\ 2\end{array} \end{array} \begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}51.625\\ 51.625\end{Bmatrix} \qquad \dots (2)$$

For element 2: (Nodes 2, 3)



Since all the parameters in elements (1) and element (2) are same, the finite element equation becomes

$$\begin{array}{c c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} & \\ \begin{bmatrix}0.886 & -0.7117\\ -0.7117 & 0.886\end{bmatrix} & \begin{array}{c}\mathbf{2}\\ \mathbf{3}\end{array} \end{array} \begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}51.625\\ 51.625\end{Bmatrix} \begin{array}{c}\mathbf{2}\\ \mathbf{3}\end{array} \qquad \dots (3)$$

Assemble the finite elements (2) and (3).

$$\begin{array}{c c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} & \\ \begin{bmatrix}0.886 & -0.7117 & 0\\ -0.7117 & 0.886 + 0.886 & -0.7117\\ 0 & -0.7117 & 0.886\end{bmatrix} & \begin{array}{c}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{array} \end{array} \begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}51.625\\ 51.625 + 51.625\\ 51.625\end{Bmatrix}$$

$$\underbrace{\begin{bmatrix}0.886 & -0.7117 & 0\\ -0.7117 & 0.886 + 0.886 & -0.7117\\ 0 & -0.7117 & 0.886\end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix}}_{[T]} = \underbrace{\begin{Bmatrix}51.625\\ 51.625 + 51.625\\ 51.625\end{Bmatrix}}_{[F]} \qquad \dots (4)$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.772 & -0.7117 \\ 0 & -0.7117 & 0.886 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 51.625 \\ 103.25 \\ 51.625 \end{Bmatrix}$$

Step 2: The first row of the force matrix is replaced by the known temperature at node 1, i.e., T_1.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.772 & -0.7117 \\ 0 & -0.7117 & 0.886 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 103.25 \\ 51.625 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value (From equation no.4) is multiplied by known temperature at node 1, i.e., -0.7117 × 393 = - 279.69. This value (as positive digit, i.e., 279.69) has been added to the second row of the force matrix.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.772 & -0.7117 \\ 0 & -0.7117 & 0.886 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 103.25 + 279.69 \\ 51.625 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.772 & -0.7117 \\ 0 & -0.7117 & 0.886 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 382.94 \\ 51.625 \end{Bmatrix} \qquad \ldots (5)$$

Solving equation (5),

$$\Rightarrow \quad 1.772\, T_2 - 0.7117\, T_3 = 382.94 \qquad \ldots (6)$$

$$-0.7117\, T_2 + 0.886\, T_3 = 51.625 \qquad \ldots (7)$$

Equation (7) × 2.4898,

$$-1.772\, T_2 + 2.2059\, T_3 = 128.536 \qquad \ldots (8)$$

$$Equation\ (6) \Rightarrow \ 1.772\, T_2 - 0.7117\, T_3 = 382.94$$

$$1.4942\, T_3 = 511.476$$

$$\Rightarrow \quad T_3 = 342.31\ K$$

Substitute T_3 value in equation (6),

$$\Rightarrow \quad 1.772(T_2) - 0.7117 \times 342.31 = 382.94$$

$$\Rightarrow \quad T_2 = 353.59\ K$$

Result: Temperature distribution:

$T_1 = 393$ K

$T_2 = 353.59$ K

$T_3 = 342.31$ K

Verification: Since the length of the fin is 50 mm, it is treated as short fin. Assume end is insulated.

Temperature distribution for short fin, end insulated is given by

(9)⇒ $$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cos h\ m\,[l - l]}{\cos h\,(m\,l)}$$

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{1}{\cos h\,(m\,l)} \qquad \ldots (10)$$

where $$m = \sqrt{\frac{h\ P}{k\ A}}$$

$\mathrm{P} = \text{Perimeter} = 2\ \times \mathrm{l}$ (Approximately)

$$= 2\ \times 0.050 = 0.1\,m$$

A = Area = Length × Thickness

$$= 0.050 \times 0.007$$

$$A = 35 \times 10^{-4}\ \mathrm{m}^2$$

$$m = \sqrt{\frac{h\ P}{k\ A}}$$

$$= \sqrt{\frac{140\ \times 0.1}{55 \times 3.5\ \times 10^{-4}}}$$

$$m\ = 26.96\,m^{-1}$$

(10)$\Rightarrow$ $$\frac{T - T_\infty}{T_b - T_\infty} = \frac{1}{\cos h\,(26.96 \times 0.050)}$$

$$\Rightarrow \quad \frac{T - 295}{393 - 295} = \frac{1}{2.05} \qquad [\because \text{Base Temperature } T_b = T_1 = 393\,K]$$

$$T - 295 = 47.8$$

$$T = 342.8\,K$$

Temperature at the end of the fin.

$$T_{x=l} = T_3 = 342.8\,K$$

(ii) Temperature at the middle of the fin:

Put $x = \frac{1}{2}$ in equation (9),

(9)$\Rightarrow$ $$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cos h\, m\,[l - \frac{l}{2}]}{\cos h\,(m\,l)}$$

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cos h\, 26.96[0.050 - \frac{0.050}{2}]}{\cos h\,(26.96 \times 0.050)} \qquad [\because T_b = T_1]$$

$$\Rightarrow \quad \frac{T - 295}{393 - 295} = \frac{1.234}{2.049}$$

$$\frac{T - 295}{393 - 295} = 0.6025$$

$$T = 354.04\,K$$

Temperature at the middle of the fin,

$$T_{x = l/2}\ T_2 = 354.04\,K$$

Example 5.4

A steel rod of diameter d = 2 cm, length L = 5 cm and thermal conductivity k= 50 W/m °C is exposed at one end to a constant temperature of 320°C. The other end is in ambient air of temperature 20°C with a convection coefficient of h=100 W/m^2 ^{0}C. Determine the temperature at the midpoint of the rod.

Given:

Diameter, d = 2 cm = 0.02 m

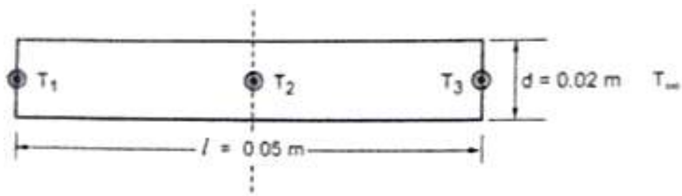
Length, l = 5 mm = 0.50 m

Thermal conductivity, k = 50 W/m^{0}C

One end temperature, $T_1 = 320^0 C + 273 = 593\ K$

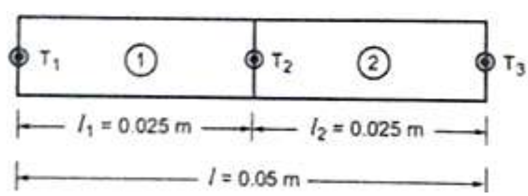
Thermal air conductivity, $T_\infty = 20^0 C + 273 = 293\ K$

Convection coefficient, h = 100 W/m^2 ^{o}C

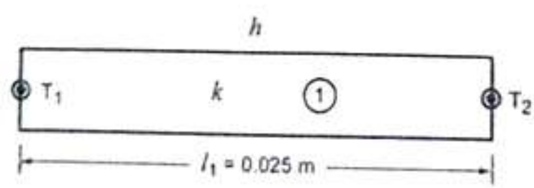


To find: Temperature at the mid point of the rod (T_2).

Solution: Discretize the rod into two equal size element.



For element 1: (Nodes 1, 2)



We know that, Finite element equation is

$$\frac{A\,k}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Q\,A\,l_1 + P\,h\,T_\infty\,l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \ldots (1)$$

where $\quad \mathrm{P} = \text{Perimeter} = \pi\,\mathrm{d} = \pi \times 0.02$

$$P = 0.0628\,m$$

$$A = Area = \frac{\pi}{4}\, d^2$$

$$= \frac{\pi}{4}\,(0.02)^2$$

$$A = 3.14 \times 10^{-4} m^2$$

Heat generation Q is not given. So, neglect that term, $\left(\frac{Q\,A\,l_1}{2}\right)$

$$(1) \Rightarrow \frac{A\,k}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{P\,h\,T_\infty\,l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Substitute A, P, k, h, l_l and T_∞ values,

$$(1) \Rightarrow \frac{3.14 \times 10^{-4} \times 50}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{100 \times 0.0628 \times 0.025}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

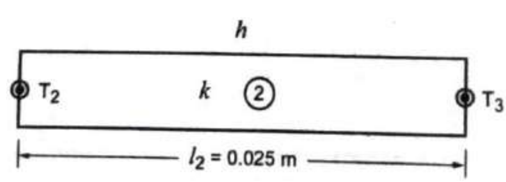
$$= \frac{0.0628 \times 100 \times 293 \times 0.025}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \left(0.628\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 0.0262\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = 23\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 0.628 & -0.628 \\ -0.628 & 0.628 \end{bmatrix} + \begin{bmatrix} 0.0524 & 0.0262 \\ 0.0262 & 0.0524 \end{bmatrix}\right)\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 23 \end{Bmatrix}$$

$$\Rightarrow \begin{array}{cc} \mathbf{1} & \mathbf{2} \\ \multicolumn{2}{c}{\begin{bmatrix} 0.6804 & -0.6018 \\ -0.6018 & 0.6804 \end{bmatrix}} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 235 \end{Bmatrix} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \qquad \ldots (2)$$

For element 2: (Nodes 2, 3)



Since all the parameters in elements (1) and element (2) are same, the finite element equation becomes

$$\Rightarrow \begin{array}{cc} \mathbf{2} & \mathbf{3} \\ \multicolumn{2}{c}{\begin{bmatrix} 0.6804 & -0.6018 \\ -0.6018 & 0.6804 \end{bmatrix}} \end{array} \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 23 \end{Bmatrix} \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \qquad \ldots (3)$$

Assemble the finite elements (2) and (3).

$$\begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 0.6804 & -0.6018 & 0 \\ -0.6018 & 0.6804+0.6804 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 23+23 \\ 23 \end{Bmatrix} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array}$$

$$\underbrace{\begin{bmatrix} 0.6804 & -0.6018 & 0 \\ -0.6018 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}_{[T]} = \underbrace{\begin{Bmatrix} 23 \\ 46 \\ 23 \end{Bmatrix}}_{[F]} \qquad \text{... (4)}$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 23 \\ 46 \\ 23 \end{Bmatrix}$$

Step 2: The first row of the force matrix is replaced by the known temperature at node 1, i.e., T_1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 593 \\ 46 \\ 23 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value (From equation no.4) is multiplied by known temperature at node 1, i.e., -0.7117 × 393 = - 279.69. This value (as positive digit, i.e., 356.867) has been added to the second row of the force matrix.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 593 \\ 146+356.867 \\ 23 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3608 & -0.6018 \\ 0 & -0.6018 & 0.6804 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 593 \\ 402.867 \\ 23 \end{Bmatrix} \qquad \text{... (5)}$$

Solving equation (5),

$$\Rightarrow \qquad 1.3608\, T_2 - 0.6018\, T_3 = 402.867 \qquad \text{... (6)}$$

$$-0.6018\,T_2 + 0.6804\,T_3 = 23 \qquad \ldots(7)$$

Equation (6) × 1.1306,

$$1.5385\,T_2 - 0.6804\,T_3 = 455.485 \qquad ..(8)$$

$$Equation\ (7) \Rightarrow -0.6018\,T_2 + 0.6804\,T_3 = 23$$

$$0.9367\,T_2 = 478.485$$

$$\Rightarrow \qquad T_2 = 510.819\,K$$

Result: Temperature at the midpoint of the rod, $T_2 = 510.819\,K$

Example 5.5

Calculate the temperature distribution in a one dimension fin with physical properties given in Fig.(i). The fin is rectangular in shape and is 120 mm long. 40 mm wide and 10 mm thick. Assume that convection heat loss occurs from the end of the fin. Use two elements. Take k = 0.3 W/mm°C; h=1×10^{-3} W/mm²°C, T_∞=20°C

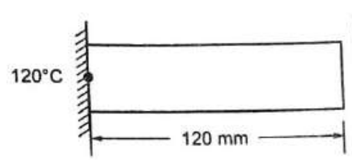


Fig. (i)

Given data:

Length, $l = 120$ mm $= 0.050$ m

Wide, $W = 40$ mm $= 0.040$ m

Thickness, $t = 10$ mm $= 0.010$ m

Thermal conductivity, $K = 0.3$ W/mm°C

$$= 0.3 \times 10^3\, W/m^o\, C = 300\ W/m^o\, C$$

Heat transfer coefficient, $h = 1 \times 10^{-3}\, W/mm^{2^o}\, C$

$$= 1 \times 10^{-3} \times 10^6\, W/m^{2^o}\, C$$

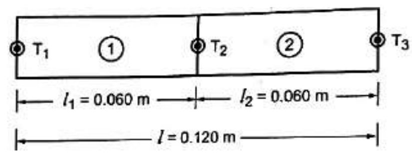
$$h = 1000\, W/m^{2^o}\, C$$

Ambient temperature, $T_\infty = 20^0C + 273 = 293$ K

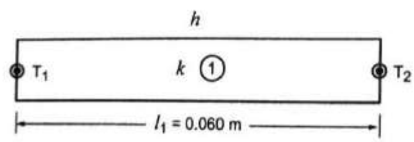
One end temperature, $T_1 = 120^0C + 273 = 393$ K

To find: Temperature distribution, $(T_1, T_2 \; and \; T_3)$

Solution: Discretize the fin into two equal size element.



For element 1: (Nodes 1, 2):



We know that, Finite element equation is

$$\frac{A\,k}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Q\,A\,l_1 + P\,h\,T_\infty\,l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{...(1)}$$

where $P = \text{Perimeter} = 2\;(W + t)$ (for rectangular)

$= 2\,[\,0.040 + 0.010]$

$P = 0.1\,m$

$A = Area = Width \times Thickness$

$= 0.040 \times 0.010$

$A = 4 \times 10^{-4} m^2$

Heat generation Q is not given. So, neglect that term, $\left(\frac{Q\,A\,l_1}{2}\right)$

$$(1) \Rightarrow \quad \frac{A\,k}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\,P\,l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{P\,h\,T_\infty\,l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Substitute A, P, k, h, l_1 and T_∞ values in equation (1).

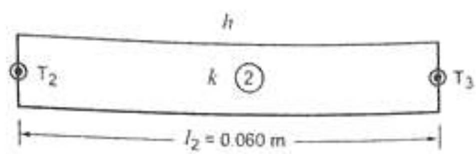
$$\Rightarrow \quad \frac{4\times 10^{-4}\times 300}{0.060}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{1000\times 0.1\times 0.060}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}$$

$$= \frac{0.1\times 1000\times 293\times 0.060}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

$$\Rightarrow \quad \left(\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} + \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}879\\ 879\end{Bmatrix}$$

$$\Rightarrow \quad \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix} \end{array} \begin{array}{c}\mathbf{1}\\ \mathbf{2}\end{array} \begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}879\\ 879\end{Bmatrix} \begin{array}{c}\mathbf{1}\\ \mathbf{2}\end{array} \qquad \ldots(2)$$

For element 2: (Nodes 2, 3)



Since all the parameters in elements (1) and element (2) are same, the finite element equation becomes

$$\Rightarrow \quad \begin{array}{c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix} \end{array} \begin{array}{c}\mathbf{2}\\ \mathbf{3}\end{array} \begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}879\\ 879\end{Bmatrix} \begin{array}{c}\mathbf{2}\\ \mathbf{3}\end{array} \qquad \ldots(3)$$

Assemble the finite elements (2) and (3).

$$\begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix}4 & -1 & 0\\ -1 & 4+4 & -1\\ 0 & -1 & 4\end{bmatrix} \end{array} \begin{array}{c}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{array} \begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}879\\ 879+879\\ 879\end{Bmatrix} \begin{array}{c}\mathbf{1}\\ \mathbf{2}\\ \mathbf{3}\end{array}$$

$$\underbrace{\begin{bmatrix}4 & -1 & 0\\ -1 & 4+4 & -1\\ 0 & -1 & 4\end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix}}_{[T]} = \underbrace{\begin{Bmatrix}879\\ 1758\\ 879\end{Bmatrix}}_{[F]} \qquad \ldots(4)$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 879 \\ 1758 \\ 879 \end{Bmatrix}$$

Step 2: The first row of the force matrix is replaced by the known temperature at node 1, i.e., T_1.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 1758 \\ 879 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value (From equation no.4) is multiplied by known temperature at node 1, i.e., -1 × 393 = - 393. This value (as positive digit, i.e., 393) has been added to the second row of the force matrix.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 393 + 1758 \\ 879 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 393 \\ 2151 \\ 879 \end{Bmatrix} \qquad \ldots (5)$$

Solving equation (5),

$$\Rightarrow \; 8\,T_2 - T_3 = 2151 \qquad \ldots(6)$$

$$-\,T_2 + 4\,T_3 = 879 \qquad ..(7)$$

Equation (7) × 8,

$$\Rightarrow -\,8\,T_2 + 32\,T_3 = 7032 \qquad ..(8)$$

$$Equation\ (6) \;\Rightarrow 8\,T_2 - T_3 = 2151$$

$$31\;T_3 = 9183$$

$$\Rightarrow \qquad T_3 = 296.226\,K$$

Substitute T_3 value in equation (6),

$$\Rightarrow 8\,(T_2) - 296.226 = 2151$$

$$\Rightarrow \qquad T_2 = 305.90\,K$$

Result: Temperature distribution:

$T_1 = 393$ K

$T_2 = 305.90$ K

$T_3 = 296.226$ K

5.6 HEAT TRANSFER IN 2-DIMENSION (THERMAL PROBLEMS)

5.6.1 Shape Function Derivation for Heat Transfer in 2D Element

We begin this section with the development of the shape function for a basic two dimensional triangular element. We consider this triangular element because its derivation is the simplest among the available two dimensional elements.

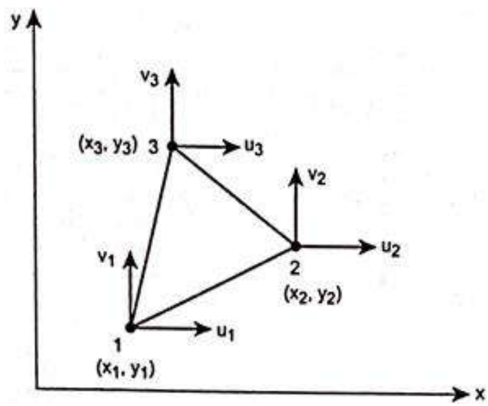


Fig 5.6 Three noded triangular element

Consider a typical triangular element with nodes 1, 2 and 3 as shown in Fig. 5.16. Let the nodal displacements be $u_1, u_2, u_3, v_1, v_2 and\ v_3$.

Displacement $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$

Since the triangular element has got two degrees of freedom at each node (u, v), the total degrees of freedom is 6. Hence it has 6 generalized coordinates.

Let , $u = \propto_1 + \propto_2 x + \propto_3 y$

$v = \propto_4 + \propto_5 x + \propto_6 y$

Where, $\propto_1, \propto_2, \propto_3, \propto_4, \propto_5$ *and* $\propto_6$ are global or generalized co-ordinates.

$$\Rightarrow \qquad u_1 = \propto_1 + \propto_2 x_1 + \propto_3 y_1$$

$$u_2 = \propto_1 + \propto_2 x_2 + \propto_3 y_2$$

$$u_3 = \propto_1 + \propto_2 x_3 + \propto_3 y_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$Let \qquad D = \begin{bmatrix} + & - & + \\ 1 & x_1 & y_1 \\ - & + & - \\ 1 & x_2 & y_2 \\ + & - & + \\ 1 & x_3 & y_3 \end{bmatrix}$$

We know, $\quad D^{-1} = \dfrac{C^T}{|D|}$

Find the co-factors of matrix D.

$$c_{11} = + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2\, y_3 - x_3 y_2)$$

$$c_{12} = - \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$c_{13} = + \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = (x_3 - x_2)$$

$$c_{21} = - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -(x_1\, y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3$$

$$c_{22} = + \begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$c_{23} = -\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$c_{31} = +\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$c_{32} = -\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} = -(y_2 - y_1) = y_1 - y_2$$

$$c_{33} = +\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$\Rightarrow \quad C = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{vmatrix}$$

$$\Rightarrow \quad C^T = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix}$$

We know that, $D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$|D| = 1\,(x_2 y_3 - x_3 y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)$$

Subsitute $C^T$and D values in equaiton

$$\Rightarrow \quad D^{-1} = \frac{1}{(x_2 y_3 - x_3 y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)} \times \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{vmatrix}$$

Subsitute D^{-1} value in equaiton

$$\Rightarrow \quad \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix} = \frac{1}{(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)}$$

$$\times \begin{vmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{vmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The area of the triangle can be expressed as a function of the x, y co-ordinates of the nodes 1, 2 and 3.

$$\Rightarrow \qquad A = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$|A| = \frac{1}{2}[1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)]$$

$$\Rightarrow \qquad 2\,A = (x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)$$

Substitute 2A values in equation

$$\Rightarrow \qquad \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix} = \frac{1}{2A} \begin{vmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{vmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix} = \frac{1}{2A} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$where, a_1 = x_2y_3 - x_3y_2; a_2 = x_3y_1 - x_1y_3; a_3 = x_1y_2 - x_2y_1$

$b_1 = y_2 - y_3; b_2 = y_3 - y_1; b_3 = y_1 - y_2$

$c_1 = x_3 - x_2; c_2 = x_1 - x_3; c_3 = x_2 - x_1$

$u \; = \propto_1 + \propto_2 + \propto_3 \, y$

We can write this equation in matrix form.

$$u = [\; l \; x \; y] \begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{Bmatrix}$ value,

$$\Rightarrow \qquad u = [\, l \; x \; y \,] \times \frac{1}{2A} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [\, l \; x \; y \,] \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [a_1 + b_1\, x + \; c_1 y \;\; a_2 + b_2 x + c_2 y \;\; a_3 \; + b_3 x + c_3 y] \;\; \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[\because (1 \times 3) \times (3 \times 3) = 1 \times 3\,]$$

$$u = \left[\frac{a_1 + b_1\, x + \; c_1 y}{2A} \;\; \frac{a_2 + b_2 x + c_2 y}{2A} \;\; \frac{a_3 + b_3 x + c_3 y}{2A} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

$$u = \; [N_1 \;\; N_2 \;\; N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Similarly, $\qquad v = \; [N_1 \;\; N_2 \;\; N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$

Where, shape function, $N_1 = \dfrac{a_1 + b_1\, x + \; c_1 y}{2A}$

$$N_2 = \frac{a_2 + b_2\, x + \; c_2 y}{2A}$$

$$N_3 = \frac{a_3 + b_3\, x + \; 3y}{2A}$$

Assemble the equation in matrix form,

Displacement funciton, $u = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$

5.7 STIFFNESS MATRIX AND LOAD VECTOR FOR HEAT TRANSFER IN TWO-DIMENSIONAL ELEMENT

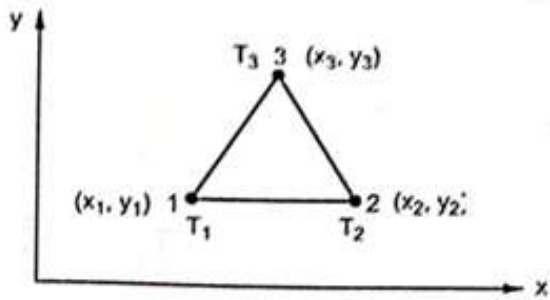


Fig. 5.7 Triangular element with nodal temperature

Triangular element is the basic element for solution of two-dimensional heat transfer problems. Consider the three-noded triangular element with nodal temperatures T_1, T_2 and T_3 as shown in Fig 5.7.

The temperature function is given by,

$$T(x, y) = N_1\, T_1 + N_2 T_2 + N_3 T_3$$

We know that,

$$\left.\begin{aligned} \text{shape functions, } N_1 &= \frac{a_1 + b_1\, x + \; c_1 y}{2A} \\ N_2 &= \frac{a_2 + b_2\, x + \; c_2 y}{2A} \\ N_3 &= \frac{a_3 + b_3\, x + \; c_3 y}{2A} \end{aligned}\right\}$$

We know that,

Stiffness matrix $\qquad [\mathrm{K_C}] = \displaystyle\int [B]^T\, [D]\, [B]\, dv$

We know that

$$\text{Strain} - \text{Displacement matrix, } \; [B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \end{bmatrix}$$

By partial differentiation,

$$\left.\begin{aligned} \frac{\partial N_1}{\partial x} = \frac{b_1}{2A} \quad \frac{\partial N_1}{\partial x} = \frac{c_1}{2A} \\ \frac{\partial N_2}{\partial x} = \frac{b_2}{2A} \quad \frac{\partial N_2}{\partial x} = \frac{c_2}{2A} \\ \frac{\partial N_3}{\partial x} = \frac{b_3}{2A} \quad \frac{\partial N_3}{\partial x} = \frac{c_3}{2A} \end{aligned}\right\}$$

Substitute the equation

$$[B] = \begin{bmatrix} \frac{b_1}{2A} & \frac{b_2}{2A} & \frac{b_3}{2A} \\ \frac{c_1}{2A} & \frac{c_2}{2A} & \frac{c_3}{2A} \end{bmatrix}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$[B]^T = \frac{1}{2A}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}$$

We know that,

$$\text{Stress} - \text{strain matrix}, [D] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$$

Assuming a unit thickness, the elements volume can be expressed as dv = Da

$$\Rightarrow \qquad [K_C\,] = \int \frac{1}{2A}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \times \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} dA$$

$$= \frac{1}{4\,A^2}\int \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} dA$$

$$= \frac{1}{4\,A^2}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\int dA$$

$$= \frac{1}{4\,A}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= \frac{1}{4\,A}\begin{bmatrix} b_1\,k_x + 0 & 0 + c_1\,k_y \\ b_2\,k_x + 0 & 0 + c_2\,k_y \\ b_3\,k_x + 0 & 0 + c_3\,k_y \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$[K_C\,] = \frac{1}{4\,A}\begin{bmatrix} (b_1^2\,k_x + c_1^2\,k_y) & (b_1 b_2 k_x + c_1 c_2 k_y) & (b_1\,b_3\,k_x + c_1\,c_3\,k_y) \\ (b_1 b_2\,k_x + c_1 c_2 k_y) & (b_2^2\,k_x + c_2^2\,k_y) & (b_2 b_3 k_x + c_2\,c_3\,k_y) \\ (b_1 b_3 k_x + c_1 c_3 k_y) & (b_2\,b_3\,k_x + c_2\,c_3\,k_y) & (b_3^2\,k_x + c_3^2\,k_y) \end{bmatrix}$$

For an isotropic material with $k_x = k_y = k$,

Stiffness matrix for conduction,

$$[K_C\,] = \frac{1}{4\,A}\begin{bmatrix} (b_1^2\; + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1\,b_3\; + c_1\,c_3\,) \\ (b_1 b_2\; + c_1 c_2) & (b_2^2\; + c_2^2\,) & (b_2 b_3 + c_2\,c_3\,) \\ (b_1 b_3 + c_1 c_3) & (b_2\,b_3\; + c_2\,c_3\,) & (b_3^2\; + c_3^2\,) \end{bmatrix}$$

To determine the stiffness matrix for convection,

$$[K_h\,] = \int_S h\,[\,N\,]^T\,[\,N\,]\,ds$$

$$= h\int \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}[\,N_1\;\;N_2\;\;N_3]\,ds$$

$$= h\int \begin{bmatrix} N_1^2 & N_1 N_2 & N_1 N_3 \\ N_1 N_2 & N_2^2 & N_2 N_3 \\ N_1 N_3 & N_2 N_3 & N_3^2 \end{bmatrix} ds$$

Let the edge 1-2 of element lies on the boundary as shown in Fig. 5.8. So that $N_3 = 0$ along this edge.

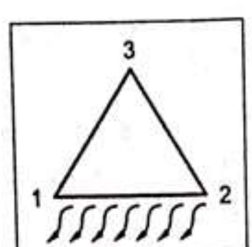


Fig. 5.8 Heat loss by convection from sides 1-2

Substitute $N_3 = 0$ in equation

$$[K_h]\,h = \int \begin{bmatrix} N_1^2 & N_1N_2 & 0 \\ N_1N_2 & N_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} ds$$

subsitute $\mathrm{N_1 = L_1, N_2 = L_2}$ and $\mathrm{N_3 = l_3}$, along the edge $1 - 2, \mathrm{N_3 = l_3 = 3}$.

Hence, $\Rightarrow \quad [K_h] = h \int_{s_1}^{s_2} \begin{bmatrix} L_1^2 & L_1L_2 & 0 \\ L_1L_2 & L_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} ds$

Where, s- denotes the direction along the edge 1-2.

We know that $\int L_1^{\propto} L_2^{\beta}\, ds = \dfrac{\alpha!\ \beta!}{(\alpha + \beta + 1)!} s$

Therefore, $\int L_1^2\, ds = \dfrac{2!}{(2+1)!} s$

$$= \frac{1 \times 2}{1 \times 2 \times 3} s$$

$$\int L_1^2\, ds = \frac{s}{3}$$

Similarly, $\int L_1\, L_2\, ds = \dfrac{1!\ 1!}{(1+1+1)!} s$

$$= \frac{1}{1 \times 2 \times 3} s$$

$$\int L_1L_2\, ds = \frac{s}{6}$$

Similarly, $\int L_2^2\, ds = \dfrac{2!}{(2+1)!} s$

$$= \frac{1 \times 2}{1 \times 2 \times 3} s$$

$$\int L_2^2\, ds = \frac{s}{3}$$

$$\Rightarrow \qquad [K_h]_{1-2} = h_{1-2}\begin{bmatrix} \frac{s_{1-2}}{3} & \frac{s_{1-2}}{6} & 0 \\ \frac{s_{1-2}}{6} & \frac{s_{1-2}}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[Direction along the edge (1-2)]

$$[K_h]_{1-2} = \frac{h_{1-2}\, s_{1-2}}{6}\begin{bmatrix} \frac{s_{1-2}}{3} & \frac{s_{1-2}}{6} & 0 \\ \frac{s_{1-2}}{6} & \frac{s_{1-2}}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, consider the edge 2-3 of element lies on the boundary

Hence, $\mathrm{N}_1 = \mathrm{L}_1 = 0, \mathrm{N}_2 = \mathrm{L}_2, \mathrm{N}_3 = \mathrm{L}_3$.

Substitute the N_1, N_2 *and* N_3 values in equation we get

$$[K_h]_{2-3} = h\int_{s_2}^{s_3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & L_2^2 & L_2L_3 \\ 0 & L_2L_3 & L_3^2 \end{bmatrix} ds$$

$$[K_h]_{2-3} = h_{2-3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{s_{2-3}}{3} & \frac{s_{2-3}}{6} \\ 0 & \frac{s_{2-3}}{6} & \frac{s_{2-3}}{3} \end{bmatrix}$$

[Direction along the edge (2-3)]

$$= \frac{h_{2-3}\, s_{2-3}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Similarly, let the edge 3 -1 of elements lies on the boundary.

Hence, $\mathrm{N}_1 = \mathrm{L}_1, \mathrm{N}_2 = \mathrm{L}_2, \mathrm{N}_3 = \mathrm{L}_3$.

Substitute the N_1, N_2 *and* N_3 values in equation

$$[K_h]_{3-1} = h\int_{s_3}^{s_1}\begin{bmatrix} L_1^2 & 0 & L_1L_3 \\ 0 & 0 & 0 \\ L_1L_3 & 0 & L_3^2 \end{bmatrix} ds$$

$$[K_h]_{3-1} = h_{3-1}\begin{bmatrix} \frac{s_{3-1}}{3} & 0 & \frac{s_{3-1}}{6} \\ 0 & 0 & 0 \\ \frac{s_{3-1}}{6} & 0 & \frac{s_{3-1}}{3} \end{bmatrix}$$

$$[K_h]_{3-1} = \frac{h_{3-1}\, s_{3-1}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Stiffness matrix for convection,

$$[\,K_h] = [K_h]_{1-2} + [K_h]_{2-3} + [K_h]_{3-1}$$

$$[\,K_h] = \frac{h_{1-2}\, s_{1-2}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{h_{2-3}\, s_{2-3}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \frac{h_{3-1}\, s_{3-1}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Stiffness matrix for 2 –dimensional heat transfer element is given by,

$$[\,K\,] = \;[\,K_C] + [\,K_h]$$

$$[\,K\,] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2\, c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2\, c_3) & (b_3^2 + c_3^2) \end{bmatrix} + \frac{h_{1-2}\, s_{1-2}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{h_{2-3}\, s_{2-3}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \frac{h_{3-1}\, s_{3-1}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

5.8 FORCE VECTOR OR LOAD VECTOR, (F)

The force vector for 2 – dimensional heat transfer element is given by

$$[F_1] = \int q_0 [\,N]^T\, dv = q_0 \int \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}\; dA$$

$$[F_1] = q_0 \int \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA$$

$$= q_0 \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA \;\; [\because dv = dA\ (unit\ thickness)]$$

By using area co- ordinates system,

$$\int L_1^{\propto}\ L_2^{\beta}\, L_3^{\gamma}\ dA = \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A$$

We know that, $\int L_1\ dA = \frac{1!}{(1+2)!} \times 2A = \frac{1}{1 \times 2\ \times 2} \times 2A$

$$\int L_1\ dA = \frac{A}{3}$$

Similarly, $\int L_2\ dA = \frac{1!}{(1+2)!} \times 2A = \frac{1}{1 \times 2\ \times 3} \times 2A$

$$\int L_2\ dA = \frac{A}{3}$$

Similarly, $\int L_3\ dA = \frac{1!}{(1+2)!} \times 2A = \frac{1}{1 \times 2\ \times 3} \times 2A$

$$\int L_3\ dA = \frac{A}{3}$$

Substitute the equations and values in equation

$$\{F_1\} = \frac{q_0 A}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$Similarly,$ $\{F_2\} = \int_{S_2} q\ [N]^T\ ds$

$$= \int q \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} ds$$

If the edge 1-2 lies on s_2,

subsitute $\mathrm{N_1 = L_1, N_2 = L_2}$ and $\mathrm{N_3 = l_3}$, along the edge $1-2, \mathrm{N_3 = L_3} = 0$.

$$\{F_2\} = q \int_{s_1}^{s_2} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} ds$$

By using surface edges,

$$\int L_1^{\propto}\ L_2^{\beta}\ \ ds = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}s$$

We know that,

$$\int L_1\ \ ds = \frac{1!}{(1+1)!}s = \frac{1}{2!}s = \frac{s}{2}$$

$similarly,$ $$\int L_2\ \ ds = \frac{1!}{(1+1)!}s = \frac{s}{2}$$

$$\{F_2\} = \frac{q_{1-2}\, s_{1-2}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

Similarly, the vector {F3} can be obtained as,

$$\{F_3\} = \int h\, t_{\infty}[N]^T\, ds$$

If the edge 1-2 lies on s3,

subsitute $\mathrm{N_1 = L_1, N_2 = L_2}$ and $\mathrm{N_3 = L_3}$, along the edge $1-2\ \ N_3 = L_3 = 0$.

$$\{F_3\} = h\, T_{\infty} \int \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} ds$$

$$= h\, T_{\infty} \int \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} ds$$

$$= h\, T_{\infty} \int \begin{Bmatrix} L_1 \\ L_2 \\ 0 \end{Bmatrix} ds$$

$$\{F_3\} = \frac{h_{1-2}\, T_\infty s_{1-2}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

5.9. SOLVED PROBLEMS [TWO DIMENSIONAL]

Example 5.6

Find the temperature distribution in a square region with uniform energy generation as shown in Fig.(i). Assume that there is no temperature variation in z-direction. Take k=30 W/cm °C, length = 10 cm, T=50°C, q= 100 W/cm³.

Given:

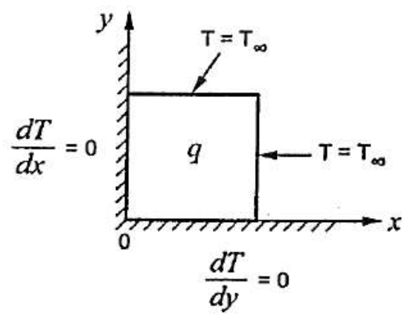


Fig. (i)

k = 30 W /Cm°C; l = 10 cm; $T_\infty = 50^{O}$c; q = 100W/cm3

To find: Temperature distribution

Solution:

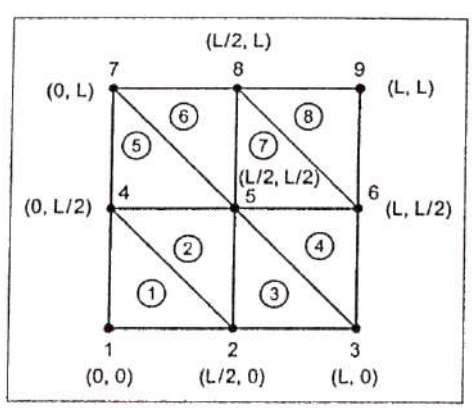
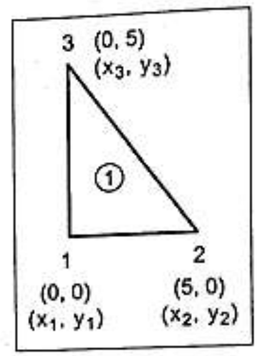


Fig. (ii)

Element / Node	1	2	3	4	5	6	7	8
1	1	4	2	5	4	7	5	8
2	2	2	3	3	5	5	6	6
3	4	5	5	6	7	8	8	9

Element (1):



$$Area\ of\ triangle\ A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 5 \end{vmatrix}$$

$$= 12.5 \text{ cm}^2$$

Length of edge 2 -3,

$$L_{2-3} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$= \sqrt{(5-0)^2 + (0-5)^2}$$

$$= 7.07 \text{ cm}$$

Stiffness matrix of element (1),

$$[K_1] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$b_1 = (y_2 - y_3) = 0 - 5 = -5$$

$$b_2 = (y_3 - y_1) = 5 - 0 = 5$$

$$b_3 = (y_1 - y_2) = 0 - 0 = 0$$

$$c_1 = (x_3 - x_2) = 0 - 5 = -5$$

$$c_2 = (x_1 - x_3) = 0 - 0 = 0$$

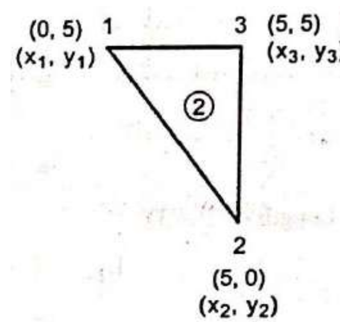
$$c_3 = (x_2 - x_1) = 5 - 0 = 5$$

Substituting the above values in [K_1],

$$[K_1] = \frac{30}{4 \times 12.5}\begin{bmatrix} 50 & -25 & -25 \\ -25 & 25 & 0 \\ -25 & 0 & 25 \end{bmatrix}$$

$$[K_1] = \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{4} \end{array} \\ \begin{bmatrix} 30 & -15 & -15 \\ -15 & 15 & 0 \\ -15 & 0 & 15 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{4} \end{array}$$

Element (2):



Area of triangle $A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 5 \\ 1 & 5 & 0 \\ 1 & 5 & 5 \end{vmatrix}$$

A = 12.5 cm^2

Length of edge 1 -2,

$$L_{1-2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(5-0)^2 + (0-5)^2}$$

$$= 7.07 \ cm$$

Stiffness matrix of element (2),

$$[K_2] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$b_1 = (y_2 - y_3) = 0 - 5 = -5$$

$$b_2 = (y_3 - y_1) = 5 - 5 = 0$$

$$b_3 = (y_1 - y_2) = 5 - 0 = 5$$

$$c_1 = (x_3 - x_2) = 5 - 5 = 0$$

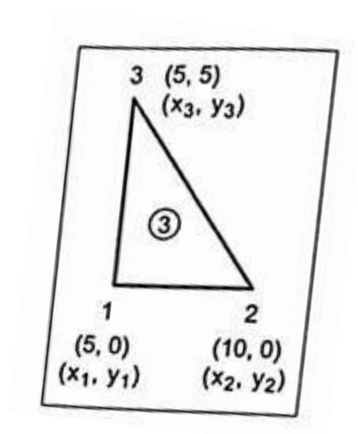
$c_2 = (x_1 - x_3) = 0 - 5 = -5$

$c_3 = (x_2 - x_1) = 5 - 0 = 5$

Substituting the above values in $[K_2]$,

$$[K_2] = \begin{array}{c} \begin{matrix} \mathbf{4} & \mathbf{2} & \mathbf{5} \end{matrix} \\ \begin{bmatrix} 15 & 0 & -15 \\ 0 & 15 & -15 \\ -15 & -15 & 30 \end{bmatrix} \end{array} \begin{matrix} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \end{matrix}$$

Element (3):



Area of triangle $\quad A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 0 \\ 1 & 10 & 0 \\ 1 & 5 & 5 \end{vmatrix}$$

$= 12.5 \text{ cm}^2$

Length of edge (2 -3),

$$L_{2-3} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$= \sqrt{(5 - 10)^2 + (5 - 0)^2}$$

$= 7.07$ cm

Stiffness matrix of element (3),

$$[K_3] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$b_1 = (y_2 - y_3) = 0 - 5 = -5$

$b_2 = (y_3 - y_1) = 5 - 0 = 5$

$b_3 = (y_1 - y_2) = 0 - 0 = 0$

$c_1 = (x_3 - x_2) = 5 - 10 = -5$

$c_2 = (x_1 - x_3) = 5 - 5 = 0$

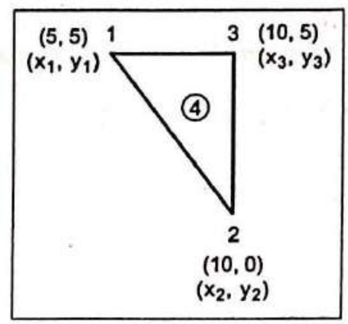
$$c_3 = (x_2 - x_1) = 10 - 5 = 5$$

Substituting the above values in [K3],

$$[K_3] = \frac{30}{4 \times 12.5}\begin{bmatrix} 50 & -25 & -25 \\ -25 & 25 & 0 \\ -25 & 0 & 25 \end{bmatrix}$$

$$[K_3] = \begin{array}{c} \begin{array}{ccc} \mathbf{2} & \mathbf{3} & \mathbf{5} \end{array} \\ \begin{bmatrix} 30 & -15 & -15 \\ -15 & 15 & 0 \\ -15 & 0 & 15 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{5} \end{array}$$

Element (4):



Area of triangle $A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 5 \\ 1 & 10 & 0 \\ 1 & 10 & 5 \end{vmatrix}$$

$$= 12.5 \text{ cm}^2$$

Length of edge (1 -2),

$$L_{1-2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(10-5)^2 + (0-5)^2}$$

$$= 7.07 \text{ cm}$$

Stiffness matrix of element (4),

$$[K_4] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$b_1 = (y_2 - y_3) = 0 - 5 = -5$$

$$b_2 = (y_3 - y_1) = 5 - 5 = 0$$

$$b_3 = (y_1 - y_2) = 5 - 0 = 5$$

$$c_1 = (x_3 - x_2) = 10 - 10 = 0$$

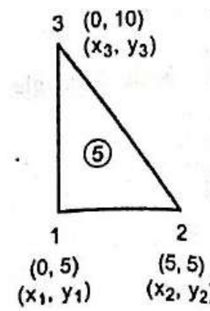
$$c_2 = (x_1 - x_3) = 5 - 10 = -5$$

$$c_3 = (x_2 - x_1) = 10 - 5 = 5$$

Substituting the above values in [K4],

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{5} & \mathbf{3} & \mathbf{6} \end{array} \\ [K_4] = & \begin{bmatrix} 15 & 0 & -15 \\ 0 & 15 & -15 \\ -15 & -15 & 30 \end{bmatrix} \begin{array}{c} \mathbf{5} \\ \mathbf{3} \\ \mathbf{6} \end{array} \end{array}$$

Element (5):



Area of triangle $A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 5 \\ 1 & 5 & 5 \\ 1 & 0 & 10 \end{vmatrix}$$

$= 12.5$ cm^2

Length of edge (2 -3),

$$L_{2-3} = \sqrt{(x_2 - x_2)^2 + (y_3 - y_2)^2}$$

$$= \sqrt{(0-5)^2 + (10-5)^2}$$

$= 7.07$ cm

Stiffness matrix of element (5),

$$[K_5] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$b_1 = (y_2 - y_3) = 5 - 10 = -5$$

$$b_2 = (y_3 - y_1) = 10 - 5 = 5$$

$$b_3 = (y_1 - y_2) = 5 - 5 = 0$$

$$c_1 = (x_3 - x_2) = 0 - 5 = -5$$

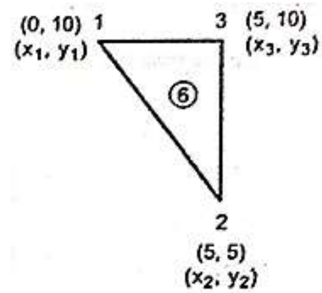
$$c_2 = (x_1 - x_3) = 0 - 0 = 0$$

$$c_3 = (x_2 - x_1) = 5 - 0 = 5$$

Substituting the above values in [K5],

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{4} & \mathbf{5} & \mathbf{7} \end{array} \\ [K_5] = & \begin{bmatrix} 30 & -15 & -15 \\ -15 & 15 & 0 \\ -15 & 0 & 15 \end{bmatrix} \begin{array}{c} \mathbf{4} \\ \mathbf{5} \\ \mathbf{7} \end{array} \end{array}$$

Element (6):



Area of triangle $A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 10 \\ 1 & 5 & 5 \\ 1 & 5 & 10 \end{vmatrix}$$

$= 12.5\ \text{cm}^2$

Length of edge (1 -2),

$$L_{1-2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(5 - 0)^2 + (5 - 10)^2}$$

$= 7.07$ cm

Stiffness matrix of element (6),

$$[K_6] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$b_1 = (y_2 - y_3) = 5 - 10 = -5$$

$$b_2 = (y_3 - y_1) = 10 - 10 = 0$$

$$b_3 = (y_1 - y_2) = 10 - 5 = 5$$

$$c_1 = (x_3 - x_2) = 5 - 5 = 0$$

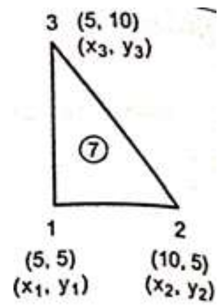
$$c_2 = (x_1 - x_3) = 0 - 5 = -5$$

$$c_3 = (x_2 - x_1) = 5 - 0 = 5$$

Substituting the above values in [K₆],

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{7} & \mathbf{5} & \mathbf{8} \end{array} \\ [K_6] = & \begin{bmatrix} 15 & 0 & -15 \\ 0 & 15 & -15 \\ -15 & -15 & 30 \end{bmatrix} \begin{array}{c} \mathbf{7} \\ \mathbf{5} \\ \mathbf{8} \end{array} \end{array}$$

Element (7):



Area of triangle $A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 5 \\ 1 & 10 & 5 \\ 1 & 5 & 10 \end{vmatrix}$$

$= 12.5\ cm^2$

Length of edge (2 -3),

$$L_{2-3} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$= \sqrt{(5 - 10)^2 + (10 - 5)^2}$$

$= 7.07$ cm

Stiffness matrix of element (7),

$$[K_7] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$b_1 = (y_2 - y_3) = 5 - 10 = -5$$

$$b_2 = (y_3 - y_1) = 10 - 5 = 5$$

$$b_3 = (y_1 - y_2) = 5 - 5 = 0$$

$$c_1 = (x_3 - x_2) = 5 - 10 = -5$$

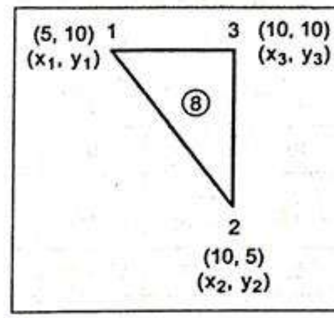
$$c_2 = (x_1 - x_3) = 5 - 5 = 0$$

$$c_3 = (x_2 - x_1) = 10 - 5 = 5$$

Substituting the above values in [K7],

$$[K_7] = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{5} & \mathbf{6} & \mathbf{8} \end{array} \\ \begin{bmatrix} 30 & -15 & -15 \\ -15 & 15 & 0 \\ -15 & 0 & 15 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{8} \end{array}$$

Element (8):



Area of triangle $A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 10 \\ 1 & 10 & 5 \\ 1 & 10 & 10 \end{vmatrix}$$

$= 12.5 \text{ cm}^2$

Length of edge (1 -2),

$$L_{1-2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(10 - 5)^2 + (5 - 10)^2}$$

$= 7.07$ cm

Stiffness matrix of element (8),

$$[K_8] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$b_1 = (y_2 - y_3) = 5 - 10 = -5$

$b_2 = (y_3 - y_1) = 10 - 10 = 0$

$b_3 = (y_1 - y_2) = 10 - 5 = 05$

$c_1 = (x_3 - x_2) = 10 - 10 = 0$

$c_2 = (x_1 - x_3) = 5 - 10 = -5$

$c_3 = (x_2 - x_1) = 10 - 5 = 5$

Substituting the above values in [K_8],

$$[K_8] = \begin{array}{c} \begin{array}{ccc} \mathbf{8} & \mathbf{6} & \mathbf{9} \end{array} \\ \begin{bmatrix} 15 & 0 & -15 \\ 0 & 15 & -15 \\ -15 & -15 & 30 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{8} \\ \mathbf{6} \\ \mathbf{9} \end{array}$$

Assemble the stiffness matrix,

[K] =

1	2	3	4	5	6	7	8	9	
30	-15	0	-15	0	0	0	0	0	1
-15	15 + 15 + 30	-15	0	-15 -15	0	0	0	0	2
0	-15	15 + 15	0	0+0	-15	0	0	0	3
-15	0	0	15 + 15 + 30	-15- 15	0	-15	0	0	4
0	-15 - 15	0+0	-15 - 15	15+15+ 15+30+ 30+15	-15- 15	0 + 0	-15 - 15	0	5
0	0	-15	0	-15 - 15	30 + 15 +15	0	0 + 0	-15	6
0	0	0	-15	0 + 0	0	15 +15	-15	0	7
0	0	0	0	-15 - 15	0 + 0	-15	30 + 15 + 15	-15	8
0	0	0	0	0	-15	0	- 15	30	9

$$[K] = \begin{array}{c} \begin{array}{ccccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \end{array} \\ \begin{bmatrix} 30 & -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ -15 & 60 & -15 & 0 & -30 & 0 & 0 & 0 & 0 \\ 0 & -15 & 30 & 0 & 0 & -15 & 0 & 0 & 0 \\ -15 & 0 & 0 & 60 & -30 & 0 & -15 & 0 & 0 \\ 0 & -30 & 0 & -30 & 120 & -30 & 0 & -30 & 0 \\ 0 & 0 & -15 & 0 & -30 & 60 & 0 & 0 & -15 \\ 0 & 0 & 0 & -15 & 0 & 0 & 30 & -15 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & -15 & 60 & -15 \\ 0 & 0 & 0 & 0 & 0 & -15 & 0 & -15 & 30 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{array}$$

Load vector for element (1),

$$\{F\} = \frac{qA}{2}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \qquad \{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{array}{c} \mathbf{1} \\ \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{4} \end{array}$$

Similarly, Load vector for element (2),

$$\{F\} = \begin{Bmatrix} F_4 \\ F_2 \\ F_5 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_4 \\ F_2 \\ F_5 \end{Bmatrix} = \begin{array}{c} \mathbf{1} \\ \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \end{array} \begin{array}{c} 4 \\ 2 \\ 5 \end{array}$$

Similarly, Load vector for element (3),

$$\{F\} = \begin{Bmatrix} F_2 \\ F_3 \\ F_5 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \\ F_5 \end{Bmatrix} = \begin{array}{c} \mathbf{1} \\ \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \end{array} \begin{array}{l} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{5} \end{array}$$

Similarly, Load vector for element (4),

$$\begin{Bmatrix} F_5 \\ F_3 \\ F_6 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_5 \\ F_3 \\ F_6 \end{Bmatrix} = \begin{array}{c} \mathbf{1} \\ \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \end{array} \begin{array}{l} \\ \mathbf{5} \\ \mathbf{3} \\ \mathbf{6} \end{array}$$

Similarly, Load vector for element (5),

$$\begin{Bmatrix} F_4 \\ F_5 \\ F_7 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_4 \\ F_5 \\ F_7 \end{Bmatrix} = \begin{array}{c} \mathbf{1} \\ \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \end{array} \begin{array}{l} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{7} \end{array}$$

Similarly, Load vector for element (6),

$$\begin{Bmatrix} F_7 \\ F_5 \\ F_8 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

1

$$\begin{Bmatrix} F_7 \\ F_5 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \begin{matrix} \mathbf{7} \\ \mathbf{5} \\ \mathbf{8} \end{matrix}$$

Similarly, Load vector for element (7),

$$\begin{Bmatrix} F_5 \\ F_6 \\ F_8 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

1

$$\begin{Bmatrix} F_5 \\ F_6 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \begin{matrix} \mathbf{5} \\ \mathbf{6} \\ \mathbf{8} \end{matrix}$$

Similarly, Load vector for element (8),

$$\begin{Bmatrix} F_8 \\ F_6 \\ F_9 \end{Bmatrix} = \frac{qA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{100 \times 12.5}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

1

$$\begin{Bmatrix} F_8 \\ F_6 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} 416.6 \\ 416.6 \\ 416.6 \end{Bmatrix} \begin{matrix} \mathbf{8} \\ \mathbf{6} \\ \mathbf{9} \end{matrix}$$

Assemble the load vector or force matrix,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \begin{array}{c} \mathbf{1} \\ \begin{Bmatrix} 416.6 \\ 1249.8 \\ 833.2 \\ 1249.8 \\ 2499.6 \\ 1249.8 \\ 833.2 \\ 1249.8 \\ 416.8 \end{Bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{array}$$

Globl matrix, $\{\mathrm{F}\} = [\mathrm{K}]\{\mathrm{T}\}$

$$\begin{Bmatrix} 416.6 \\ 1249.8 \\ 833.2 \\ 1249.8 \\ 2499.6 \\ 1249.8 \\ 833.2 \\ 1249.8 \\ 416.8 \end{Bmatrix} = \begin{array}{c} \begin{array}{ccccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \end{array} \\ \begin{bmatrix} 30 & -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ -15 & 60 & -15 & 0 & -30 & 0 & 0 & 0 & 0 \\ 0 & -15 & 30 & 0 & 0 & -15 & 0 & 0 & 0 \\ -15 & 0 & 0 & 60 & -30 & 0 & -15 & 0 & 0 \\ 0 & -30 & 0 & -30 & 120 & -30 & 0 & -30 & 0 \\ 0 & 0 & -15 & 0 & -30 & 60 & 0 & 0 & -15 \\ 0 & 0 & 0 & -15 & 0 & 0 & 30 & -15 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & -15 & 60 & -15 \\ 0 & 0 & 0 & 0 & 0 & -15 & 0 & -15 & 30 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{array} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}$$

Applying boundary conditions,

$$T_1 = T_1, T_2 = T_2, T_3 = 50^0 C, T_4 = T_4, T_5 = T_5,$$

$$T_6 = 50^0 C, \qquad T_7 = 50^0 C, T_8 = 50^0 C, T_9 = 50^0 C.$$

$$\Rightarrow \qquad T_3 = T_6 = T_7 = T_8 = T_9 = 50^0 C$$

$$\begin{Bmatrix} 416.6 \\ 1249.8 \\ 833.2 \\ 1249.8 \\ 2499.6 \\ 1249.8 \\ 833.2 \\ 1249.8 \\ 416.8 \end{Bmatrix} = \begin{array}{c} \begin{array}{ccccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \end{array} \\ \begin{bmatrix} 30 & -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ -15 & 60 & -15 & 0 & -30 & 0 & 0 & 0 & 0 \\ 0 & -15 & 30 & 0 & 0 & -15 & 0 & 0 & 0 \\ -15 & 0 & 0 & 60 & -30 & 0 & -15 & 0 & 0 \\ 0 & -30 & 0 & -30 & 120 & -30 & 0 & -30 & 0 \\ 0 & 0 & -15 & 0 & -30 & 60 & 0 & 0 & -15 \\ 0 & 0 & 0 & -15 & 0 & 0 & 30 & -15 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & -15 & 60 & -15 \\ 0 & 0 & 0 & 0 & 0 & -15 & 0 & -15 & 30 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{array} \begin{Bmatrix} T_1 \\ T_2 \\ 50 \\ T_4 \\ T_5 \\ 50 \\ 50 \\ 50 \\ 50 \end{Bmatrix}$$

We know that,

$$\Rightarrow \quad 30T_1 - 15\,T_2 - 15\,T_4 = 416.4 \qquad \ldots(1)$$

$$\Rightarrow \quad -15T_1 + 60\,T_2 - 15\,(50) - 30\,T_5 = 1249.8$$

$$-15T_1 + 60\,T_2 - 30T_5 = 1999.8 \qquad \ldots(2)$$

$$\Rightarrow \quad -15T_1 + 30\,(50) - 15(50) = 833.2 \qquad \ldots(3)$$

$$\Rightarrow \quad -15T_1 + 60\,T_4 - 30T_5 \; - 15(50) = 1249.8 \qquad \ldots(4)$$

$$\Rightarrow -30T_2 - 30T_4 + 120\,T_5 - 30(50) - 30(50) = 2499.6 \qquad \ldots(5)$$

$$\Rightarrow \quad -15(50) - 30T_5 + 60(50) - 15(50) = 1249.8 \qquad \ldots(6)$$

$$\Rightarrow \quad -15T_4 + 30(50) - 15(50) = 833.2 \qquad \ldots(7)$$

$$\Rightarrow \quad -30T_5 - 15(50) + 60(50) - 15(50) = 1249.8 \qquad \ldots(8)$$

Equaiton (7)$\Rightarrow \quad -15\;\mathrm{T}_4 = 833.2 + 15\,(50) - 30(50)$

$$T_4 = 119.4^O C$$

Substitute the values T_4 and T_5 in equation (4),

$$\Rightarrow -15T_1 - 30(119.4) + 120(105.6) - 30(50) - 15(50) = 1249.8$$

$$-15T_1 = 1249.8 + 30(119.4) - 120(105.6) + 30(50) - 15(50)$$

$$T_1 = \;\; 133.31^O C$$

Substitute the values T_1 and T_4 in equation (1),

$$\Rightarrow 30(133.3) - 15\,T_2 - 15\,(119.4) = 416.6$$

$$T_2 = 119.42^O C$$

Result: Temperature distribution,

$$T_1 = 133.31^O C$$

$$T_2 = 119.42^O C$$

$$T_3 = 50^O C$$

$T_4 = 119.42^O C$

$T_5 = 105.6^O C$

$T_6 = 50^O C$

$T_7 = 50^O C$

$T_8 = 50^O C$

$T_9 = 50^O C$

Example 5.7

Compute the element matrix and vectors for the element shown in Fig.(i) when the edges 2-3 and 3-1 experience convection heat loss.

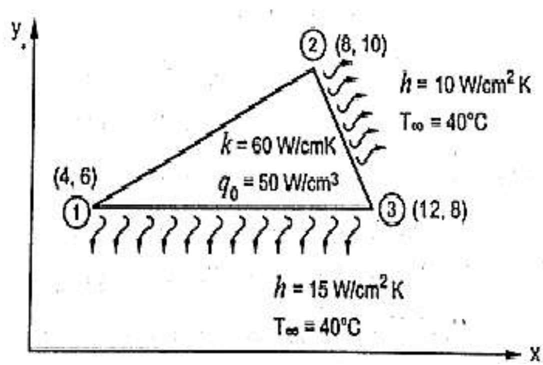


Fig. (i).

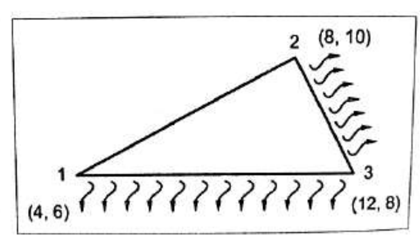
Given: k = 60 W/cm K

$q_0 = 50$ W/cm^3

$(x_1, y_1) = 4, 6$

$(x_2, y_2) = 8, 10$

$(x_3, y_3) = 12, 8$



To find: (1) Element matrix [K_C] and [K_h] (2) force vector

Solution: We know that, Stiffness matrix for conduction,

$$[K_C] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

Where,

$$\text{Area of triangle } A, = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 4 & 6 \\ 1 & 8 & 10 \\ 1 & 12 & 8 \end{vmatrix}$$

A = 12.5 cm^2

Where,

$$b_1 = (y_2 - y_3) = (10 - 8) = 2$$

$$b_2 = (y_3 - y_1) = (8 - 6) = 2$$

$$b_3 = (y_1 - y_2) = (6 - 10) = -4$$

$$c_1 = (x_3 - x_2) = (12 - 8) = 4$$

$$c_2 = (x_1 - x_3) = (4 - 12) = -8$$

$$c_3 = (x_2 - x_1) = (8 - 4) = 4$$

Substituting the above values in [K$_C$],

$$[K_C] = \frac{60}{4 \times 12}\begin{bmatrix} (4+16) & (4-32) & (-8+16) \\ (4-32) & (4+64) & (-8-32) \\ (-8+16) & (-8-32) & (16+16) \end{bmatrix}$$

$$[K_C] = \begin{bmatrix} 25 & -35 & 10 \\ -35 & 85 & -50 \\ -10 & -50 & 40 \end{bmatrix} \quad ..(1)$$

We know that, stiffness matrix for convection,

$$[K_h] = \frac{h_{1-3}\, s_{1-3}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \frac{h_{3-2}\, s_{3-2}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$where, s_{1-3} = Length\ of\ edge\ 1-3$

$$s_{1-3} = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$$

$$s_{1-3} = \sqrt{(4 - 12)^2 + (6 - 8)^2}$$

$$s_{1-3} = 8.25\ cm$$

Similarly, s_{3-2} = Length of edge $3-2$

$$s_{3-2} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$s_{3-2} = \sqrt{(12-8)^2 + (8-10)^2}$$

$$s_{3-2} = 4.47\ cm$$

Substitution of these values in equation [K_h]

$$[K_h] = \frac{15 \times 8.25}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \frac{10 \times 4.47}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[K_h] = \begin{bmatrix} 41.25 & 0 & 20.62 \\ 0 & 14.90 & 7.45 \\ 20.62 & 7.45 & 56.15 \end{bmatrix} \quad \ldots(2)$$

Force vector, $= \dfrac{q_0 A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ at side s_1

$$= \frac{50 \times 12}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\{F_1\} = \begin{Bmatrix} 200 \\ 200 \\ 200 \end{Bmatrix} \quad \ldots(3)$$

$$\{F_2\} = \frac{q\ s_{2-1}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}\ at\ side\ s_2.$$

$$\{F_2\} = 0 \quad \ldots(4)$$

[∵ No boundary heat flux is specified]

$$\{F_3\} = \frac{h_{3-2\,T_\infty} s_{3-2}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + \frac{h_{1-3\,T_\infty} s_{1-3}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\{F_3\} = \frac{10 \times 40 \times 4.47}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} + \frac{15 \times 40 \times 8.25}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$$

$$\{F_3\} = \begin{Bmatrix}2475\\894\\3369\end{Bmatrix}$$

Result:

(1) Element matrix:

$$\text{Conduciton: } [\mathrm{K_C}] = \begin{bmatrix}25 & -35 & 10\\-35 & 85 & -50\\10 & -50 & 40\end{bmatrix}$$

$$\text{Conveciton: } [\mathrm{K_h}] = \begin{bmatrix}41.25 & 0 & 20.62\\0 & 14.90 & 7.45\\20.62 & 7.45 & 56.15\end{bmatrix}$$

$$\text{(2)Force vector: } \{\mathrm{F_1}\} = \begin{Bmatrix}200\\200\\200\end{Bmatrix}$$

$$\{F_2\} = 0$$

$$\{F_3\} = \begin{Bmatrix}2475\\894\\3369\end{Bmatrix}$$

Example 5.8

Find the temperature distribution in a square region with uniform energy generation as shown in Fig.(i). Assume that there is no temperature variation in the z-direction. Take k=30 W/cm °C, 1= 10 cm, T=50°C, q = 100 W/cm³.

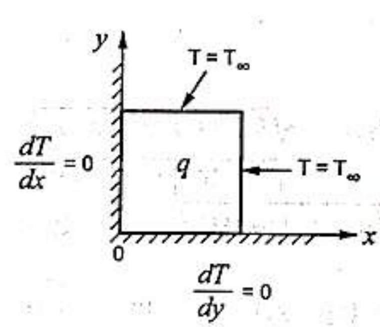


Fig. (i).

Given: k = 30 W/cm ^{O}C

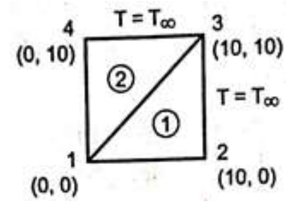
l = 10 cm

$T_\infty = 50^0$C

Q = 100 W/cm^3

To find: Temperature distribution.

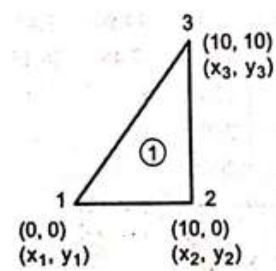
Solution: Converting into 2 elements



Element 1:

$$\text{Area of triangle, } A_1 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 10 & 10 \end{vmatrix}$$

$$A_1 = 50\ cm^2$$



Length at side, 1-3

$$\Rightarrow \quad l_{1-3} = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$$

$$= \sqrt{(0 - 10)^2 + (10 - 0)^2}$$

$$l_{1-3} = 14.14\ cm$$

We know that,

$$b_1 = (y_2 - y_3) = (0 - 10) = -10$$

$$b_2 = (y_3 - y_1) = (10 - 0) = 10$$

$$b_3 = (y_1 - y_2) = (0 - 0) = 0$$

$$c_1 = (x_3 - x_2) = (10 - 10) = 0$$

$$c_2 = (x_1 - x_3) = (0 - 10) = -10$$

$$c_3 = (x_2 - x_1) = (10 - 0) = 10$$

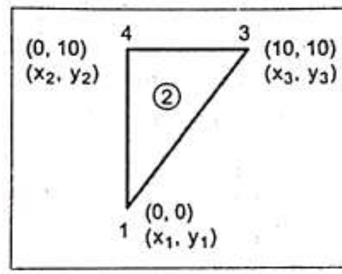
$$[K_1] = \frac{k}{4\,A}\begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix}$$

$$[K_1] = \frac{30}{4 \times 50}\begin{bmatrix} 100 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ [K_1] = & \begin{bmatrix} 15 & -15 & 0 \\ -15 & 30 & -15 \\ 0 & -15 & 15 \end{bmatrix} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \end{array}$$

Similarly,

Element 2:



Area of triangle $A_2, = \frac{1}{2} b\, h$

$$= \frac{1}{2} \times 10 \times 10$$

$$A_2 = \frac{1}{2}(100) = 50\ cm^2$$

We know that,

$$b_1 = (y_2 - y_3) = (10 - 10) = 0$$

$$b_2 = (y_3 - y_1) = (10 - 0) = 10$$

$$b_3 = (y_1 - y_2) = (0 - 10) = -10$$

$$c_1 = (x_3 - x_2) = (10 - 0) = 10$$

$$c_2 = (x_1 - x_3) = (0 - 10) = -10$$

$$c_3 = (x_2 - x_1) = (0 - 0) = 0$$

Stiffness matrix for element (2),

$$[K_2] = \frac{30}{4 \times 50}\begin{bmatrix} 100 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{1} & \mathbf{4} & \mathbf{3} \end{array} \\ [K_2] = & \begin{bmatrix} 15 & -15 & 0 \\ -15 & 30 & -15 \\ 0 & -15 & 15 \end{bmatrix} \begin{array}{c} \mathbf{1} \\ \mathbf{4} \\ \mathbf{3} \end{array} \end{array}$$

Assembling [K] matrix,

$$[K] = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{matrix} \\ \begin{bmatrix} 30 & -15 & 0 & -15 \\ -15 & 30 & -15 & 0 \\ 0 & -15 & 30 & -15 \\ -15 & 0 & -15 & 30 \end{bmatrix} \end{array} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{matrix}$$

Load vector, $\{F\} = \frac{q\,A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

Element (1),

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{100 \times 50}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1666.67 \\ 1666.67 \\ 1666.67 \end{Bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

Similarly,

Element (2),

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 1666.67 \\ 1666.67 \\ 1666.67 \end{Bmatrix}\begin{matrix} 1 \\ 4 \\ 3 \end{matrix}$$

Assembling load vector,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} 3.3 \times 10^3 \\ 1.6 \times 10^3 \\ 3.3 \times 10^3 \\ 1.6 \times 10^3 \end{Bmatrix}$$

Global matrix, $\{F\} = [K][T]$

Temperature distribution at node 2, 3 and 4 are

$$T_\infty = T_2 = T_3 = T_4 = 50^O C$$

$$\begin{Bmatrix} 3.3 \times 10^3 \\ 1.6 \times 10^3 \\ 3.3 \times 10^3 \\ 1.6 \times 10^3 \end{Bmatrix} = \begin{bmatrix} 30 & -15 & 0 & -15 \\ -15 & 30 & -15 & 0 \\ 0 & -15 & 30 & -15 \\ -15 & 0 & -15 & 30 \end{bmatrix} = \begin{Bmatrix} T_1 \\ 50 \\ 50 \\ 50 \end{Bmatrix}$$

We know that,

$$3.3 \times 10^3 = (30\,T_1) - (15 \times 50) + (0 \times 50) - (15 \times 50)$$

$$T_1 = 160^O C$$

Result: Temperature distribution,

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 160 \\ 50 \\ 50 \\ 50 \end{Bmatrix} {}^oC$$

5.10 DYNAMIC ANALYSIS

5.10.1 Introduction

It provides an elementary introduction to time-dependent problems. It provides the basic equations necessary for structural dynamics analysis and develop both the lumped and the consistent mass matrix involved in the analysis of bar, beam and spring elements.

We discuss method the formulation and solution of vibration problems using finite element

5.10.2 Fundamentals of Vibration

Any motion which repeats itself after an interval of time is called vibration or oscillation or periodic motion.

All bodies possessing mass and elasticity are capable of producing vibrations. Vibration problems, in practice, occur wherever there are rotating or moving parts in a machinery. The study of vibration is concerned with oscillatory motions of the bodies and the forces associated with them.

Illustration: Consider a spring-mass system constrained to move in a rectilinear manner along the axis of the spring, as shown in Fig.5.9.

When the mass is displaced from its equilibrium position A, the internal forces in the form of elastic or strain energy are present in the body; and hence the mass reaches position B.

At release, these forces bring the mass to its original position. At the equilibrium position A, the whole of the elastic or strain energy is converted into kinetic energy due to which the mass continues to move in the opposite direction to position C.

At C, the whole of the kinetic energy is again converted into elastic or strain energy due to which the body again returns to the equilibrium position A.

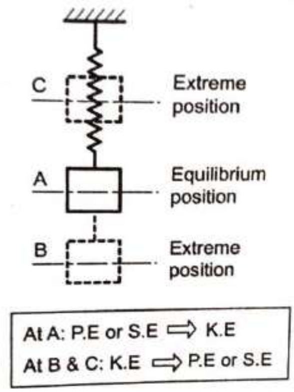


Fig. 5.9. Vibration of a spring-mass system

In this way, vibratory motion is repeated indefinitely and exchange of energy takes place.

Similarly, the swinging of simple pendulum is another example of vibration as the motion of ball is to and fro from its mean position repeatedly.

5.10.3 Causes of Vibrations:

The main causes of vibration are as follows:

1. **Unbalanced forces** in the machine. These forces are produced from within machine itself.
2. **Elastic nature** of the system.
3. **Self-excitations** produced by the dry friction between the two mating surfaces.
4. External excitations applied on the system.
5. Winds may cause the vibrations in certain systems such as transmission and telephone lines under certain conditions.
6. Earthquakes also cause vibrations and are greatly responsible for the failure of and dams, many buildings, etc.

5.10.4 Effects of Vibrations

(i) Negative effects: The existence of vibrating elements in any mechanical produces unwanted noise, high stresses, wear, poor reliability and premature failure of one more of the parts. In addition to this, vibrations are a great source of human discomfort in the form of physical and mental strains.

(ii)Positive effects: Inspite of the harmful effects, the vibratory systems are built into the system machines. Examples are almost all musical instruments, vibrating conveyors, vibrating screens, shakers, stress relievers, etc.

5.10.5 Methods of Elimination/Reduction of the Undesirable Vibrations

The undesirable vibrations can be eliminated or reduced by one or more of the following methods.

- By removing the causes of vibration.
- By resting the machinery on proper type of isolators.
- By using shock absorbers.
- By using dynamic vibration absorbers.
- By using the screens (if noise is the objection).

5.10.6. Terminology Used in Vibratory Motion

The terms commonly used in the study of vibrations are presented in

1. **Periodic motion:** A motion which repeats itself after equal interval of time.
2. **Time period (t_p):** It is the time taken by a motion to repeat itself. It is also called as period of vibration, and is measured in seconds.
3. **Cycle:** It is the motion completed during one time period.
4. **Frequency (f):** It is the number of cycles completed in one second. It is expressed in hertz (Hz). It is a reciprocal of time period. Mathematically, $f = \frac{1}{t_p}$Hz.
5. **Natural frequency:** Frequency of free vibration of the system.
6. **Amplitude (X):** The maximum displacement of a vibrating body from the mean position.
7. **Resonance:** When the frequency of the external force is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This phenomenon is known as resonance.
8. **Damping:** It is the resistance to the motion of a vibrating body.

5.10.7 Simple Harmonic Motion

- ✓ Since most of the vibrating systems follow simple harmonic motion (SHM), therefore is essential to have proper understanding of SHM related basic concepts.
- ✓ A body is said to have simple harmonic motion (SHM), if it moves or vibrates about a position such that its acceleration is always proportional to its distance from the position and is directed towards the mean position or equilibrium position.

Consider a particle 'P' moving around a circle with a uniform angular velocity o rad/s as shown in fig.5.10

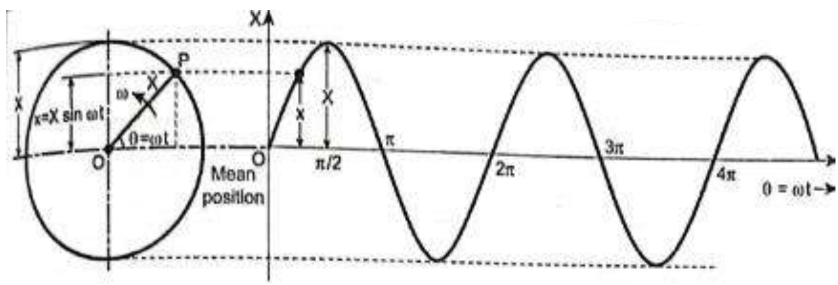


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Fig 5.10 Simple harmonic motion of a particle moving around a circle

Displacement of particle 'P' from mean position after time 't', as shown in Fig.5.10, is given by

x = X sin of

X = Maximum displacement (or amplitude) of particle from mean position.

Velocity of particle after time ‘t’ is given by

$$v = \frac{dx}{dt} - \omega X \cos \ \omega t$$

Acceleration of particle after time ‘t’ is

$$a = \frac{d^2x}{dt^2} = -\omega^2 X \sin \omega t = -\omega^2 x \qquad [\text{since } x = \mathrm{X} \sin \omega t]$$

or $$\frac{d^2x}{dt^2} \pm \omega^2 x = 0$$

The above equation is known as differential equation or fundamental equation of S.H.M

Time period and frequency:

$$\text{Time period}, t_p = \frac{2\pi}{\omega}$$

$$\text{Frequency}, f = \frac{1}{t_p} = \frac{\omega}{2\pi}$$

5.10.8 Types of Vibrations

Vibrations may be classified according to:

(a) the actuating force on the body, and

(b) the stresses in the supporting medium, as shown in Fig.5.11.

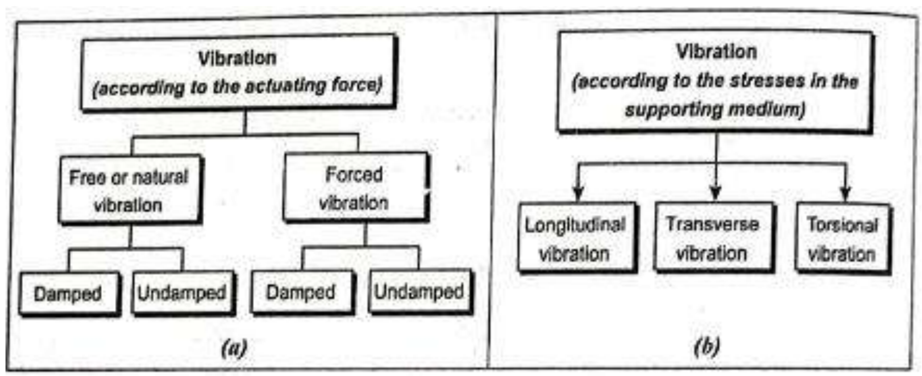

Fig. 5.11. Types of vibrations

I. According to the Actuating Force

1. *Free or Natural Vibrations*

✓ If the periodic motion continues after the cause of original disturbance (i.e., initial displacement) is removed, then the body is said to be under **free or natural vibrations**.

✓ The frequency of the free vibrations is called free or natural frequency.

✓ Example: Oscillation of a simple pendulum

2. *Forced Vibrations*

✓ When the body vibrates under the influence of external force, then the body is said to be under forced vibrations.

- ✓ The vibrations have the same frequency as the applied force.
- ✓ Examples: Vibrations in machine tools, electric bells, vibratory conveyors, etc.

3. Damped Vibrations

When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. That is, if the vibratory system has a damper, the motion of the system will be opposed by it and the energy of the system will be dissipated in friction.

On the contrary, the system having no damper is known as undamped vibration.

If the damper is connected with free vibrating body to control vibrations, then it is called free damped vibrations. If the damper is connected with forced vibrating body to control vibrations, then it is called forced damped vibrations.

Examples: Vibrations in all machinery in actual use are damped in nature.

4. Undamped vibrations

If no energy is lost or dissipated in friction or other resisting force during vibration, then such vibration is known as undamped vibration.

In other words, the system having no damper produces undamped vibrations.

In the vibratory system, if the amount of external excitation is known in magnitude, it causes deterministic vibration.

II. According to Motion of System with Respect to Axis

Consider a vibrating body, e.g., a rod, shaft or spring. Fig.5.12 shows a heavy disc carried on one end of a weightless shaft, the other end being fixed. This system can execute any one of the following types of vibrations.

1. Longitudinal Vibrations

When the particles of the shaft or disc moves parallel to the axis of the shaft, then the vibrations are known as longitudinal vibrations, as shown in Fig. 5.12 (a).

2. Transverse Vibrations

When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, then the vibrations are known as transverse vibrations, as shown in Fig. 5.12 (b).

3. Torsional Vibrations

When the particles of the shaft or disc move in a circle about the axis of the shaft, then the vibrations are known as torsional vibrations, as shown in Fig. 5.12 (c).

5.11 EQUATION OF MOTION USING LAGRANGE'S APPROACH

$$\frac{\partial}{\partial t}\left(\frac{dT}{d\dot{u}}\right) - \frac{\partial T}{\partial u} + \frac{\partial \pi}{\partial u} = \frac{F}{u}$$

Where $\frac{F}{u}$ = Generalised force in the coordinates u

$$\dot{u} = \frac{\partial u}{\partial t}$$

5.12 Formulation of Finite Element Equations

We know that, shape functions for a typical finite element, the displacement of an interior point can be written in terms of the nodal degree of freedom as,

$$\{ u\} = N_1 u_1 + N_2 u_2 \qquad \text{... (1)}$$

Differentiating with time the velocity at the point is given by,

$$\{\dot{u}\} = N_1 u_1 + N_2 u_2 \qquad \text{... (2)}$$

For the bar or truss element,

$$u = \left[\left(1 - \frac{x}{l}\right)\left(\frac{x}{l}\right)\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{... (3)}$$

$$\dot{u} = \left[\left(1 - \frac{x}{l}\right)\left(\frac{x}{l}\right)\right]\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \qquad \text{... (4)}$$

For the beam element,

$$v = [N_1\ N_2\ N_3\ N_4]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \qquad \text{... (5)}$$

$$\dot{v} = [N_1\ N_2\ N_3\ N_4]\begin{Bmatrix} \dot{v}_1 \\ \dot{\theta}_1 \\ \dot{v}_2 \\ \dot{\theta}_2 \end{Bmatrix} \qquad \text{... (6)}$$

For a two-dimensional element, each point can have *u* and *v* displacements,

$$\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{\dot{u}\} = \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \\ \dot{u}_3 \\ \dot{v}_3 \end{Bmatrix}$$

The kinetic of an element mass, m = p v within the element is given by,

$$T = \frac{1}{2} m\, v^2$$

$$dT = \frac{1}{2}(dm)\ (v)^2$$

$$dT = \frac{1}{2}\rho\ dv\ (v^2) \qquad [\because m = p\ v; dm = p\ dv]$$

$$dT = \frac{1}{2}\{\dot{u}\}^T\{\dot{u}\}\rho\ dv \qquad \text{... (7)}$$

where, $\{u\} = [\,N\,]\{u\}^e$

$$\{\dot{u}\} = [\,N\,]\{\dot{u}\}^e \qquad \text{... (8)}$$

Substitute the equation (7) in equation (8),

$$dT = \frac{1}{2}\{\dot{u}\}^{e^T}[N]^T\ [N]\ \{\dot{u}\}^e\rho\ dv \qquad \text{... (9)}$$

Integrating the above equation,

$$T = \frac{1}{2}\{\dot{u}\}^{e^T}\left(\int_v \rho[N]^T[N]\,dv\right)\{\dot{u}\}^e$$

$$T = \frac{1}{2}\{\dot{u}\}^{e^T}[m]^e\{\dot{u}\}^e \qquad \text{... (10)}$$

where, $[\mathrm{m}]^{\mathrm{e}} = \int_{\mathrm{v}} \rho[\mathrm{N}]^{\mathrm{T}}[\mathrm{N}]$ is the consistent mass matrix for the element,

The total potential energy of the system,

$$\pi_P = \frac{1}{2}\{u\}^T[K]\{u\} - \{u\}^T\{F\} \qquad \text{... (11)}$$

$$Equaiton\ (10) \Rightarrow T = \frac{1}{2}\{\dot{u}\}^T[m]\{\dot{u}\} \qquad \text{... (12)}$$

$$\Rightarrow \frac{\partial T}{\partial u} = 0,$$

$$\frac{\partial T}{\partial\{\dot{u}\}} = [m]\{\dot{u}\},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\{\dot{u}\}}\right) = [m]\{\ddot{u}\},$$

We know that,

$$\frac{\partial \pi}{\partial\{u\}} = [\,K]\{u\} - [F]$$

Substituting the above values in Lagrange's equation of motion,

$$[m]\{\ddot{u}\} + [K]\{u\} - [F] = 0$$

$$i.e., \qquad [m]\{\ddot{u}\} + [K]\{u\} = \{F\} \qquad \text{... (13)}$$

5.13. CONSISTENT MASS MATRIX FOR VARIOUS ELEMENT

For the bar element: shape Functions are,

$$N_1 = 1 - \frac{x}{l}, N_2 = \frac{x}{l}$$

Mass matrix, $$[m] = \int_v \rho [N]^T [N]\, dv \qquad \dots (14)$$

$$= \rho\, A \int_0^l \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} [N_1 \quad N_2]\, dx \qquad [\because dv = A\, dx]$$

$$= \rho\, A \int_0^l \begin{Bmatrix} \left(1 - \frac{x}{l}\right) \\ \frac{x}{l} \end{Bmatrix} \left[\left(1 - \frac{x}{l}\right) \quad \left(\frac{x}{l}\right)\right] dx$$

$$= \rho\, A \int_0^l \begin{bmatrix} \left(1 - \frac{x}{l}\right) & \frac{x}{l} - \frac{x^2}{l^2} \\ \frac{x}{l} - \frac{x^2}{l^2} & \frac{x^2}{l^2} \end{bmatrix} dx$$

$$= \rho\, A \begin{bmatrix} \dfrac{\left(1 - \frac{x}{l}\right)^3}{3\left(\frac{-1}{l}\right)} & \dfrac{x^2}{2\,l} - \dfrac{x^3}{3\,l^2} \\ \dfrac{x^2}{2\,l} - \dfrac{x^3}{3\,l^2} & \dfrac{x^2}{3\,l^2} \end{bmatrix}_0^l$$

$$= \rho\, A \begin{bmatrix} \frac{l}{3} & \frac{l}{2} - \frac{l}{3} \\ \frac{l}{2} - \frac{l}{3} & \frac{l}{3} \end{bmatrix}$$

$$= \rho\, A \begin{bmatrix} \frac{l}{3} & \frac{l}{6} \\ \frac{l}{6} & \frac{l}{3} \end{bmatrix}$$

$$[m] = \frac{\rho\, A\, l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \dots (15)$$

For the Beam Element: Shape functions are,

$$N_1 = 1 - \frac{3\,x^2}{l^2} + \frac{2\,x^3}{l^3}$$

$$N_2 = x - \frac{2\,x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3\,x^2}{l^2} - \frac{2\,x^3}{l^3}$$

$$N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

We know that

Mass matrix, $$[m] = \int_v \rho [N]^T [N]\, dv \qquad \text{... (16)}$$

$$= \rho\, A \int_0^l \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = [N_1\ N_2\ N_3\ N_4] dx \qquad \text{... (17)}$$

Substituting N_1, N_2, N_3 *and* N_4 values and after performing all the integrations, the beam element mass matrix is obtained as

$$[m] = \frac{\rho\, A\, l}{420} \begin{bmatrix} 156 & 22\,l & 54 & -13\,l \\ 22\,l & 4\,l^2 & 13\,l & -3/l^2 \\ 54 & 13\,l & 156 & -22\,l \\ -13\,l & -3\,l^2 & -22\,l & 4/l^2 \end{bmatrix} \qquad \text{.. (18)}$$

5.14 LUMPED MASS MATRIX

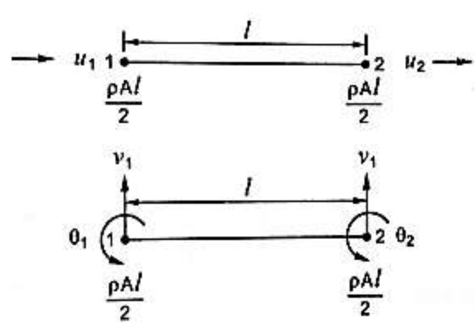

Fig. 5.13 Mass lumping for bar and beam element

The lumped mass matrix for a bar element is given by

$$[m]_{lumped} = \frac{\rho\, A\, l}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \dots (19)$$

The lumped mass matrix for a beam element is given by

$$[m]_{lumped} = \frac{\rho\, A\, l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \dots (20)$$

5.15 FORMULAE USED

1. Longitudinal Vibration of Bar

Finite element equation,

$$\{[K] - [m\,]\omega^2\,\{u\} = \{F\}$$

$$\text{Where, Stiffness matrix, }\{[\mathrm{K}] = \frac{\mathrm{E\,A}}{\mathrm{L}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$[m]_{lumped} = \frac{\rho\, A\, L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ for consistent mass matrix}$$

$$[m] = \frac{\rho\, A\, L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for lumped mass matrix}$$

2. Transverse Vibration of Beam

Finite element equation,

$$\{[K] - [m\,]\omega^2\,\{u\} = \{F\}$$

$$\text{Where, Stiffness matrix, }\{[\mathrm{K}] = \frac{\mathrm{E\,A}}{\mathrm{l}^3}\begin{bmatrix} 12 & 6\,\mathrm{l} & -12 & 6\,\mathrm{l} \\ 6\,\mathrm{l} & 4\,\mathrm{l}^2 & -6\,\mathrm{l} & 2\,\mathrm{l}^2 \\ -12 & -6\,\mathrm{l} & 12 & -6\,\mathrm{l} \\ 6\,\mathrm{l} & 2\,\mathrm{l}^2 & -6\,\mathrm{l} & 4\,\mathrm{l}^2 \end{bmatrix}$$

$$\text{Mass matrix, } \quad [\mathrm{m}] = \frac{\rho\,\mathrm{A\,L}}{420}\begin{bmatrix} 156 & 22\,\mathrm{l} & 54 & -13\,\mathrm{l} \\ 22\,\mathrm{l} & 4\,\mathrm{l}^2 & 13\,\mathrm{l} & -3\,\mathrm{l}^2 \\ 54 & 13\,\mathrm{l} & 156 & -22\,\mathrm{l} \\ -13\,\mathrm{l} & -3\,\mathrm{l}^2 & -22\,\mathrm{l} & 4\,\mathrm{l}^2 \end{bmatrix}$$

For consistent mass matrix

$$[m] = \frac{\rho\, A\, L}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; for\ lumped\ mass\ matrix$$

5.16 SOLVED PROBLMES

Example 5.9

Find the natural frequency of longitudinal vibration of the unconstrained stepped bar as shown in Fig.(i).

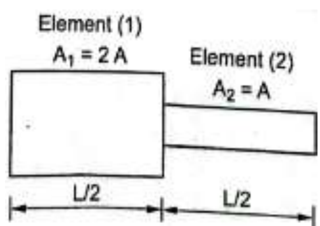

Fig. (i).

Given:

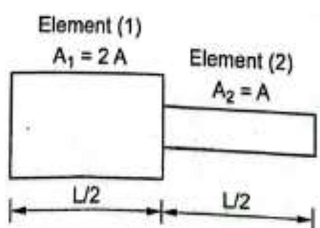

Fig. (ii).

Element (1)	**Element (2)**
$\text{Area}, A_1 = 2\text{ A}$	$\text{Area}, A_2 = 2\text{ A}$
$\text{Length}, L_1 = \frac{L}{2}$	$\text{Length}, L_2 = \frac{L}{2}$
$\text{Young's modulus}, E_1 = E$	$\text{Young's modulus}, E_2 = E$
$\text{Density}, \rho_1 = \rho$	$\text{Density}, \rho_2 = \rho$

To find: Natural frequencies of the rod.

Solution: The bar with two element and 3 nodes area as shown in Fig.(ii). The stiffness matrix of the two elements are,

$$[K_1] = \frac{A_1\, E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2\,A\,E}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{4\,A\,E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_1] = \frac{2\,A\,E}{L}\begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array}$$

Similarly, Stiffness matrix for element (2),

$$[K_2] = \frac{A_2\, E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{A\,E}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_1] = \frac{2\,A\,E}{L}\begin{array}{c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \text{... (2)}$$

Assemble the stiffness matrix,

$$[K_1] = \frac{2\,A\,E}{L}\begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \text{... (3)}$$

Mass matrix for Element (1),

$$[m_1] = \frac{\rho_1\, A_1\, L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

[From equation (15)]

$$[m_1] = \frac{\rho \times 2\, A \times \frac{L}{2}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho\, A\, L}{6} \begin{array}{cc} \begin{matrix} \mathbf{1} & \mathbf{2} \end{matrix} & \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \end{array}$$

$$[m_1] = \frac{\rho\, A\, L}{12} \begin{array}{cc} \begin{matrix} \mathbf{1} & \mathbf{2} \end{matrix} & \\ \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} & \begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \end{array} \qquad \ldots (4)$$

Similarly, mass matrix for Element (2),

$$[m_2] = \frac{\rho_2\, A_2\, L_2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho\, A\, \times \left(\frac{L}{2}\right)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_2] = \frac{\rho\, A\, L}{12} \begin{array}{cc} \begin{matrix} \mathbf{2} & \mathbf{3} \end{matrix} & \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix} \end{array} \qquad \ldots (5)$$

Assemble the mass matrix, $[m] = \frac{\rho\, A\, L}{12} \begin{array}{cc} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{matrix} & \\ \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} \end{array}$ ... (6)

Since, the bar is unconstrained (no degrees of freedom is fixed), the finite element equation is

$$\{[K] - [m]\omega^2\}\{u\} = \{P\}$$

Substitute [K] and [m] values

$$\left[\frac{2A\, E}{L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho\, A\, L}{12} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Applying boundary conditions,

$$P_1 = P_2 = P_3 = 0$$

[No degrees of freedom is fixed]

We set the determinant of the coefficient matrix equal to zero, we have

$$\left|\frac{2A\,E}{L}\begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix} - \omega^2 \frac{\rho\,A\,L}{12}\begin{bmatrix}4 & 2 & 0\\ 2 & 6 & 1\\ 0 & 1 & 2\end{bmatrix}\right| = 0 \qquad \ldots(7)$$

Divide both sides by $\left(\frac{2A\,E}{L}\right)$

$$\left|\begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix} - \frac{\omega^2 \frac{\rho\,A\,L}{12}}{\frac{2A\,E}{L}}\begin{bmatrix}4 & 2 & 0\\ 2 & 6 & 1\\ 0 & 1 & 2\end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix} - \frac{\rho\,L^2\,\omega^2}{24\,A}\begin{bmatrix}4 & 2 & 0\\ 2 & 6 & 1\\ 0 & 1 & 2\end{bmatrix}\right| = 0 \qquad \ldots(8)$$

Take, $\qquad \beta^2 = \frac{\rho\,L^2\,\omega^2}{24\,E}$

Equation (8) can be rewritten as,

$$\left|\begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix} - \beta^2\begin{bmatrix}4 & 2 & 0\\ 2 & 6 & 1\\ 0 & 1 & 2\end{bmatrix}\right| = 0$$

$$\Rightarrow \qquad \begin{bmatrix}2(1-2\beta^2) & -2(1-\beta^2) & 0\\ -2(1+\beta^2) & 3(1-2\,\beta^2) & -(1+\beta^2)\\ 0 & -1(1+\beta^2) & (1-2\,\beta^2)\end{bmatrix} = 0$$

$$\Rightarrow \qquad 2(1-2\beta^2)[3\,(1-2\,\beta^2)^2-(1+2\,\beta^2)^2]$$
$$+\,2(1+\beta^2)[\,-2(1+\beta^2)(1-2\,\beta^2)] = 0$$

Bu simplifying the above equation, we get

$$\Rightarrow \qquad 18[\beta^2(1-2\beta^2)(\beta^2-\ 2) = 0 \qquad \ldots(9)$$

The roots of equation (9) give the natural frequencies of the bar.

We know that, $\qquad \beta^2 = \frac{\rho\,L^2\omega^2}{24\,E}$

when, $\qquad \beta^2 = 0 \Rightarrow \omega_1^2 = 0 \Rightarrow \omega_1 = 0$

when, $\qquad \beta^2 = \frac{1}{2} \Rightarrow \omega_2^2 = \frac{12\,E}{\rho\,L^2} \Rightarrow \omega_2 = 3.46\,\left[\frac{E}{(\rho\,L^2)}\right]^{\frac{1}{2}}\ rad/s$

when, $\beta^2 = 2 \Rightarrow \omega_3^2 = \frac{48\,E}{\rho\,L^2} \Rightarrow \omega_3 = 6.92\left[\frac{E}{(\rho\,L^2)}\right]^{\frac{1}{2}}\ rad/s$

$\therefore Natural\ frequencies\ are, \omega_1 = 0$

$$\omega_2 = 3.46\left[\frac{E}{(\rho\,L^2)}\right]^{\frac{1}{2}}$$

$$\omega_3 = 6.92\left[\frac{E}{(\rho\,L^2)}\right]^{\frac{1}{2}}$$

Result: Natural frequencies of longitudinal vibration,

$$\omega_1 = 0$$

$$\omega_2 = 3.46\left[\frac{E}{(\rho\,L^2)}\right]^{\frac{1}{2}}\ rad/s$$

$$\omega_3 = 6.92\left[\frac{E}{(\rho\,L^2)}\right]^{\frac{1}{2}}\ rad/s$$

Example 5.11

For the bar as shown in Fig.(i) with length 2 L, modulus of elasticity E, mass density p, and cross-sectional area A, determine the first two natural frequencies.

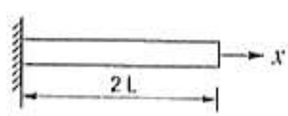

Fig. (i).

Given:

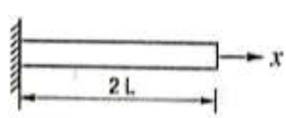

Fig. (ii).

$\text{Length, L} = 2\ \text{L}$

$\text{Young's modulus, E} = \text{E}$

Mass Density, $\rho = \rho$

Cross-sectional area, A = A

To find: Natural frequencies.

Solution: We can divide the bar with two as shown in Fig.(iii).

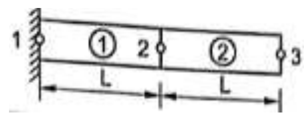

Fig. (iii).

Stiffness matrix for element (1):

$$[K_1] = \frac{A\,E}{L} \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

Similarly,

$$\textbf{\textit{Element}}\ (\mathbf{2}):\quad [K_2] = \frac{A\,E}{L} \begin{array}{c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{2} \\ \mathbf{3} \end{array}$$

Assembling the element matrix,

$$[K] = \frac{A\,E}{L} \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \ldots (1)$$

Lumped mass matrix or consistent mass matrix can be used for solving the problem.

Lumped mass matrix for Element (1):

$$[m_1] = \frac{\rho\,A\,L}{2} \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array} \qquad [From\ equaiton\ (15)]$$

Similarly

Element (2),

$$[m_2] = \frac{\rho\,A\,L}{2} \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array}$$

Assemble the mass matrix, $[m] = \frac{\rho\, A\, L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$ ... (2)

Global matrix, for bar element,

$$\{[K] - \omega^2 [m]\}\{u\} = \{P\}$$

$$\left[\frac{A\,E}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho\, A\, L}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \qquad \text{... (3)}$$

Applying boundary conditions,

$u_1 = 0\ (fixed),$ $\qquad P_1 = 0$

$u_2 = u_2$ $\qquad P_2 = 0$

$u_3 = u_3$ $\qquad P_3 = 0$

$$\Rightarrow \qquad \left[\left[\frac{A\,E}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho\, A\, L}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\right]\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m]matrix. The final reduced equation is,

$$\left|\frac{A\,E}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho\, A\, L}{12}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right|\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0 \qquad \text{... (4)}$$

To obtain a solution to the set of homogeneous equation in equation (4), we set the determinant of the coefficient matrix equal to zero.

$$\left|\frac{A\,E}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho\, A\, L}{12}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad \text{... (5)}$$

Divide both sides by ρAL,

$$\left|\frac{A\,E}{\rho\, A\, L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\, \frac{\rho\, A\, L}{2\times \rho\, A\, L}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad [\because \lambda = \omega^2]$$

$$\left|\frac{E}{\rho\ L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

Take, $$\mu = \frac{E}{\rho \ L^2}$$

$$\left| \mu \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (2\,\mu - \lambda) & -\mu \\ -\mu & \left(\mu - \frac{\lambda}{2}\right) \end{vmatrix} = 0 \qquad \text{... (6)}$$

$$\Rightarrow \qquad \left[(2\mu - \lambda)(\,\mu - \frac{\lambda}{2})\right][-\mu^2] = 0$$

$$\Rightarrow \qquad \left(2\,\mu^2 - \mu\,\lambda - \mu\,\lambda + \frac{\lambda^2}{2} - \mu^2\right) = 0$$

$$\Rightarrow \qquad \mu^2 - 2\mu\,\lambda + \frac{\lambda^2}{2} = 0$$

$$\Rightarrow \qquad \frac{\lambda^2}{2} - 2\mu\,\lambda + \mu^2 = 0 \qquad \text{... (7)}$$

By solving the quadratic equation (7),

$$\lambda = -(-2\mu) \pm \frac{\sqrt{4\mu^2 - \frac{4}{2}\mu^2}}{\left(\frac{2}{2}\right)} \qquad [\because \lambda = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}]$$

$$\lambda = 2\mu \pm \mu\sqrt{2}$$

$$\lambda = \mu\,[2 \pm \sqrt{2}\,]$$

$$\therefore \lambda_1 = 3.41\,\mu, \lambda_2 = 0.585\,\mu,$$

$Natural\ frequencies\ are,$

we know that $\lambda = \omega^2 = 0$

$\Rightarrow \omega = \sqrt{\lambda}$

$\Rightarrow \omega_1 = \sqrt{3.41\,\mu}$

$\omega_1 = 1.85\sqrt{\mu}\ rad/s$

$\therefore \qquad \omega_1 = 1.85\sqrt{\frac{E}{\rho\ L^2}}\, rad/s \qquad \left[\because\ \mu = \frac{E}{\rho\, L^2}\right]$

Similarly, $\qquad \omega_2 = \sqrt{0.585\,\mu}$

$$\omega_2 = 0.76\sqrt{\mu} \quad rad/sec$$

$$\omega_2 = 0.76\sqrt{\frac{E}{\rho\ L^2}}\ rad/s$$

Result: Natural frequencies are,

$$\omega_1 = 1.85\sqrt{\frac{E}{\rho\ L^2}}\, rad/s$$

$$\omega_2 = 0.76\sqrt{\frac{E}{\rho\ L^2}}\, rad/s$$

Example 5.12

Consider a uniform cross-section bar as shown in Fig.(i) of length "L" made up of a material whose Young's modulus and density are given by E and ρ. Estimate the natural frequencies of axial vibration of the bar using both lumped and consistent mass matrix.

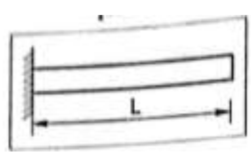

Fig. (i). A uniform bar

Given: $Length, L = L$

$Young's\ modulus, E = E$

$Density, \rho = \rho$

To find: Natural frequencies of axial vibration of the bar by using both lumped and consistent mass matrix.

Solution:

Lumped mass matrix [One element]

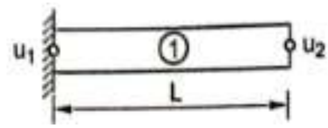

Fig. (ii).

Using just one element for the entire rod i.e., L =L and using lumped mass matrix, we have

$$\{[K] - \omega^2 [m]_{lump}\}\{u\} = 0$$

$$\{[K]\{u\} - \omega^2 [m]_{lump}\}\{u\}$$

$$\frac{A\,E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \omega_{lump}^2 \begin{bmatrix} \frac{\rho\,A\,L}{2} & 0 \\ 0 & \frac{\rho\,A\,L}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[\because [m]_{lump} = \frac{\rho\,A\,L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}]$$

Applying boundary conditions,

$u_1 = 0\ (fixed), \qquad u_2 = u_2$

$$\frac{A\,E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} - \omega_{lump}^2 \begin{bmatrix} \frac{\rho\,A\,L}{2} & 0 \\ 0 & \frac{\rho\,A\,L}{2} \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\frac{A\,E}{L}(u_2) = \omega_{lump}^2 \frac{\rho\,A\,L}{2}(u_2)$$

$$\frac{E}{L} = \omega_{lump}^2 \frac{\rho\ L}{2}$$

$Hence,$ $$\omega_{lump} = \sqrt{\frac{2\,E}{\rho\,L^2}}$$

$$\omega_{lump} = \frac{1.414}{L}\sqrt{\frac{E}{\rho}} rad/s \qquad \ldots(1)$$

Consistent mass matrix (one element):

We know that, with one element and consistent mass matrix,

$$\{[\,K\,] - \omega_{cons}^2[\,m]\}\{u\} = 0$$

$$[\,K\,]\{u\} = \omega_{cons}^2[\,m]$$

$$\frac{A\,E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \omega_{cons}^2 \frac{\rho\,A\,L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\left[\because [m]_{cons} = \frac{\rho\,A\,L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]$$

Applying boundary conditions, $u_1 = 0(fixed)$

$$u_2 = u_2$$

$$\frac{A\,E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} - \omega_{cons}^2 \frac{\rho\,A\,L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\frac{A\,E}{L}(u_2) = \omega_{cons}^2 \cdot \frac{\rho\,A\,L}{6}(2\,u_2)$$

$$\frac{E}{L} = \omega_{cons}^2 \cdot \frac{\rho\;L}{3}$$

Therefore, $$\omega_{cons} = \sqrt{\frac{3\,E}{\rho\,L^2}}$$

$$\omega_{cons} = \frac{1.732}{L}\sqrt{\frac{E}{\rho}} rad/s \qquad \ldots(2)$$

Lumped mass matrix (Two element):

Divide the rod into 2 elements as shown in Fig. (iii).

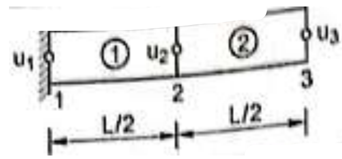

Fig. (iii)

Stiffness matrix for bar element,

Element (1):

$$[K_1] = \frac{A\,E}{L_1}\begin{array}{cc} \mathbf{1} & \mathbf{2} \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right. & \left.\right] \end{array}\begin{matrix} 1 \\ 2 \end{matrix} \qquad [\because L_1 = \frac{L}{2}]$$

$$= \frac{A\,E}{(\frac{L}{2})}\begin{array}{cc} \mathbf{1} & \mathbf{2} \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right. & \left.\right] \end{array}\begin{matrix} 1 \\ 2 \end{matrix}$$

Similarly,

$$[K_1] = \frac{2\,A\,E}{L}\begin{array}{cc} \mathbf{1} & \mathbf{2} \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right. & \left.\right] \end{array}\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix}$$

Element (2): $$[K_2] = \frac{A\,E}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{A\,E}{(\frac{L}{2})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [\because L_2 = \frac{L}{2}]$$

$$[K_2] = \frac{2\,A\,E}{L}\begin{array}{cc} \mathbf{2} & \mathbf{3} \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right. & \left.\right] \end{array}\begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix}$$

Assemble the stiffness matrix,

$$[K] = \frac{2\ A\ E}{L} \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \text{... (3)}$$

Lumped mass matrix for bar element

Element (1):

$$[m_1] = \frac{\rho\ A\ L_1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[m_1] = \frac{\rho\ A\ L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \qquad \left[\because L_1 = \frac{L}{2}\right]$$

Similarly, $$[m_1] = \frac{\rho\ A\ L_1}{4} \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array}$$

Element (2),

$$[m_2] = \frac{\rho\ A\ L_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[m_2] = \frac{\rho\ A\ L}{2} \begin{array}{c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \left[\because L_2 = \frac{L}{2}\right]$$

Assemble the lumped mass matrix,

$$[m] = \frac{\rho\ A\ L}{4} \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \text{... (4)}$$

We know that, Global matrix, for axial vibration of the bar,

$$\{[K] - \omega_{lump}^2 [m]\}\{u\} = 0 \qquad \text{... (5)}$$

$$[K]\{u\} = \omega_{lump}^2 [m]\{u\}$$

$$\frac{2\ A\ E}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \omega_{lump}^2 \frac{\rho\ A\ L}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{... (6)}$$

$$\left[\frac{2\,A\,E}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \omega^2\frac{\rho\,A\,L}{12}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} - \begin{Bmatrix}P_1\\ P_2\\ P_3\end{Bmatrix} \qquad \text{... (3}$$

Applying boundary conditions,

$$u_1 = 0\ (fixed),$$

$$u_2 = u_2$$

$$u_3 = u_3$$

$$\Rightarrow \qquad \frac{2\,A\,E}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\end{Bmatrix} = \omega_{lump}^2\frac{\rho\,A\,L}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\frac{2\,A\,E}{L}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \frac{\rho\,A\,L}{12}\omega_{lump}^2\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} \qquad \text{... (7)}$$

Divide both sides by ρ A L.

$$\frac{2\,A\,E}{\rho\,A\,L^2}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \frac{\rho\,A\,L}{4\,\rho\,A\,L}\omega_{lump}^2\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}$$

$$\left\{\frac{E}{\rho\;L^2}\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix} - \omega_{lump}^2 \times \left(\frac{1}{4}\right)\right\}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = 0 \qquad \text{... (8)}$$

To obtain a solution to the set of homogeneous equation in equation (8), we set the determinant of the coefficient matrix equal to zero.

$$\left|\frac{E}{\rho\;L^2}\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix} - \omega_{lump}^2 \times \left(\frac{1}{4}\right)\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right| = 0$$

$Take,\ \ \omega_{lump}^2 = \lambda$

$$\frac{E}{\rho\;L^2} = \mu$$

$$\left|\mu\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix} - \lambda\begin{bmatrix}0.5 & 0\\ 0 & 0.25\end{bmatrix}\right| = 0$$

$$\Rightarrow \qquad \begin{vmatrix} (4\,\mu - 5\,\lambda) & (-2\,\mu) \\ (-2\,\mu) & (2\,\mu - 0.25\,\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad [(4\,\mu - 0.5\,\lambda)(2\,\mu - 0.25\,\lambda)] - (4\,\mu^2) = 0$$

$$\left(8\,\mu^2 - \mu\,\lambda - \lambda\mu + 0.125\,\lambda^2 - 4\,\mu^2\right) = 0$$

$$0.125\,\lambda^2 - 2\mu\,\lambda + 4\,\mu^2 = 0$$

By solving the quadratic equation (7),

$$\lambda = \frac{2\mu \pm \sqrt{(-2\mu)^2 - 4(0.125)(4\,\mu^2)}}{(2X0.125)} \quad \left[\because \lambda = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}\right]$$

$$\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 2\,\mu^2}}{0.25}$$

$$\lambda = \frac{2\mu \pm \mu\sqrt{2}}{0.25}$$

$$= \frac{2\mu \pm 1.42\,\mu}{0.25}$$

$$\therefore \qquad \lambda = 8\,\mu \pm 5.68\,\mu,$$

$$\lambda_1 = 13.653\mu, \qquad \lambda_2 = 2.343\mu$$

$$\therefore \lambda_1 = \frac{13.65\,E}{\rho\,L^2} \qquad \lambda_2 = \frac{2.343}{\rho\,L^2}$$

We know that, natural frequencies are,

$$\lambda = \omega_{lump}^2$$

$$\Rightarrow \qquad \omega_{lump} = \sqrt{\lambda}$$

$$\therefore \qquad \omega_{1\,lump} = \sqrt{\frac{13.65\,E}{\rho\,L^2}} = \frac{3.695}{L}\sqrt{\frac{E}{\rho}} \quad rad/s \qquad \ldots (10)$$

Similarly, $\omega_{2\ lump} = \sqrt{\frac{2.343\ E}{\rho\ L^2}} = \frac{1.531}{L}\sqrt{\frac{E}{\rho}} \quad rad/s$ ... (11)

Consistent mass Matrix (Two Elements)

We know that,

Global stiffness matrix, $[\,\mathrm{K}] = \frac{2\ A\ E}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ [From equation no. (3)]

We know that,

Consistent mass matrix for element (1):

$$[\,m_1] = \frac{\rho\ A\ L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho\ A\ L}{12}\begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array} \qquad \left[\because L_1 = \frac{L}{2}\right]$$

Similarly,

Element (2):

$$[\,m_2] = \frac{\rho\ A\ L_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho\ A\ L}{12}\begin{array}{c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ 2 \\ 3 \end{array} \qquad \left[\because L_2 = \frac{L}{2}\right]$$

Assemble the mass matrix, [m]

$$[\,m] = \frac{\rho\ A\ L}{12}\begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array}$$

Global matrix, for axial vibration of bar,

$$\{[K] - \omega_{Con}^2 [m]\}\{u\} = 0$$

$$\left[\frac{2\,A\,E}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \omega_{con}^2\frac{\rho\,A\,L}{4}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 0\\ 0 & 1 & 2\end{bmatrix}\right]\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} = 0$$

Applying boundary conditions,

$u_1 = 0\ (fixed),$

$u_2 = u_2$

$u_3 = u_3$

$$\Rightarrow \qquad \frac{2\,A\,E}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\end{Bmatrix} = \omega_{con}^2\frac{\rho\,A\,L}{4}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\end{Bmatrix} = 0$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and m] matrix. The final reduced equation is,

$$\left[\frac{2\,A\,E}{L}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \omega_{con}^2\frac{\rho\,A\,L}{12}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = 0$$

Divide both sides by ρ A L.

$$\left[\frac{2\,A\,E}{\rho\,A\,L^2}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \omega_{con}^2\frac{\rho\,A\,L}{12\rho\,A\,L}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = 0$$

$$\left[\frac{E}{\rho\ L^2}\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix} - \omega_{con}^2\left(\frac{1}{12}\right)\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = 0 \qquad \ldots(12)$$

To obtain a solution to the set of homogeneous equation in equation (12), we set the determinant of the coefficient matrix equal to zero.

$$\left|\frac{E}{\rho\ L^2}\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix} - \omega_{con}^2\begin{bmatrix}0.333 & 0.0833\\ 0.0833 & 0.1666\end{bmatrix}\right| = 0$$

$$Take, \lambda = \omega_{con}^2\ and\ \frac{E}{\rho\,L^2} = \mu$$

$$\left|\mu\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix} - \lambda\begin{bmatrix}0.333 & 0.0833\\ 0.0833 & 0.1666\end{bmatrix}\right| = 0$$

$$\Rightarrow \qquad \begin{vmatrix}(4\,\mu - 0.333\,\lambda) & (-2\,\mu - 0.0833\,\lambda)\\ (-2\,\mu = 0.0833\,\lambda) & (2\,\mu - 0.166\,\lambda)\end{vmatrix} = 0$$

$$\Rightarrow \quad [(4\,\mu - 0.333\,\lambda)(2\,\mu - 0.166\,\lambda)] - [(-2\,\mu - 0.0833\,\lambda)(-2\,\mu - 0.833\,\lambda) = 0$$

$$\Rightarrow \quad [\left(8\,\mu^2 - 0.664\,\mu\,\lambda - 0.666\,\mu\lambda + 0.0552\,\lambda^2\right) - (4\mu^2 + 0.1666\mu\,\lambda + 0.1666\,\mu\,\lambda + 0.00693\,\lambda^2)] = 0$$

$$\Rightarrow \quad \left[4\,\mu^2 - 2.656\,\mu\,\lambda - 0.04827\,\lambda^2\right] = 0$$

$$\Rightarrow \quad -0.04827\lambda^2 - 2.656\,\mu\,\lambda + 4\,\mu^2 = 0$$

$$\Rightarrow \quad 0.04827\lambda^2 + 2.656\,\mu\,\lambda - 4\,\mu^2 = 0 \qquad \ldots (13)$$

By solving the quadratic equation (13),

$$\lambda = \frac{-2.656\mu \pm \sqrt{(2.656\,\mu)^2 - 4(0.04827)(-4\,\mu^2)}}{2(0.04827)}$$

$$\left[\because \lambda = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}\right]$$

$$= \frac{-2.656\,\mu \pm \sqrt{5.05\,\mu^2 - 0.772\,\mu^2}}{00965}$$

$$= \frac{-2.656\,\mu \pm 0.4119\,\mu}{0.0965} = 27.52\mu \pm 4.268\,\mu$$

$$\lambda = \mu\,[-27.52 \pm 4.268]$$

$$\lambda_1 = 32.69\mu \qquad \lambda_2 = 2.59\,\mu$$

Thus the natural frequencies are,

$$\lambda = \omega^2$$

$$\omega = \sqrt{\lambda}$$

$$\therefore \omega_1 = \sqrt{2.59\,\mu}$$

$$\omega_1 = 1.604\sqrt{\mu}$$

$$\omega_1 = 1.60\sqrt{\frac{E}{\rho\, L^2}}$$

$$\omega_1 = \frac{1.60}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

Similarly, $$\omega_2 = \sqrt{32.69\ \mu} = 5.71\sqrt{\mu} = 5.71\sqrt{\frac{E}{\rho\, L^2}}$$

$$\omega_2 = \frac{5.71}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

Result: Natural frequencies are,

(1) Using one element,

(a) Lumped Mass matrix

$$\omega_{lump} = \frac{1.414}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

(b) Consistent mass Matrix

$$\omega_{cons} = \frac{1.732}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

(2) Using two elements

(a) Lumped Mass matrix

$$\omega_1 = \frac{3.69}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

$$\omega_2 = \frac{1.53}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

(b) Consistent mass Matrix

$$\omega_1 = \frac{1.60}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

$$\omega_2 = \frac{5.71}{L}\sqrt{\frac{E}{\rho}}\ rad/s$$

Example 5.13

For the one-dimensional bar shown in Fig.(i), determine the natural frequencies of longitudinal vibration using two elements of equal length. Take $E = 2 \times 10^5$ N/mm^2, $\rho = 0.8 \times 10^{-4}$N/mm^3, and L = 400 mm.

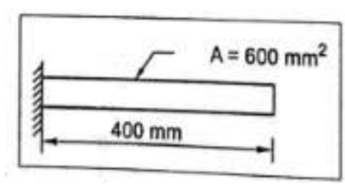

Fig. (i)

Given:

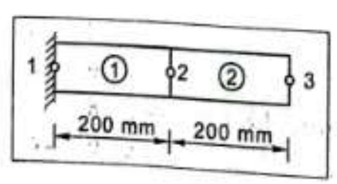

Fig. (ii)

For element (1): $L_1 = 200\ mm$

$A_1 = 600\ mm^2$

For element (1): $L_2 = 200\ mm$

$A_2 = 600\ mm^2$

$E = 2 \times 10^5 N/mm^2$

$\rho = 2 \times 10^5 N/mm^2$

To find: Natural frequencies of longitudinal bar.

Solution: the bar has 2 elements with 3 nodes s shown I Fig.(ii).

We know that,

Stiffness matrix for,

Element (1):

$$[K_1] = \frac{A_1\, E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{600\ \times 2\ \times 10^5}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_1] = 10^5 \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Similarly,

Element (2): $[K_2] = \frac{A_2\, E}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \frac{600\ \times 2\ \times 10^5}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_2] = 10^5 \begin{array}{cc} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} & \\ \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} & \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \end{array}$$

Assemble the stiffness matrix,

$$[K] = 10^5 \begin{array}{cc} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} & \\ \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \end{array} \qquad \text{... (1)}$$

We know that, mass matrix for

Element (1):

$$[m_1] = \frac{\rho\, A_1\, L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{0.8 \times 10^{-4} \times 600\ \times 200}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Similarly,
$$\begin{array}{ccc} & 1 & 2 \\ [m_1] = & \begin{bmatrix} 3.2 & 1.6 \\ 1.6 & 3.2 \end{bmatrix} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \end{array}$$

Element (2),

$$[m_2] = \frac{\rho\, A_2\, L_2}{2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{0.8 \times 10^{-4} \times 600 \times 200}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} & \mathbf{2} \quad \mathbf{3} & \\ [m_2] = & \begin{bmatrix} 3.2 & 1.6 \\ 1.6 & 3.2 \end{bmatrix} & \begin{array}{c} \mathbf{2} \\ \mathbf{3} \end{array} \end{array}$$

Assemble the mass matrix,

$$\begin{array}{ccc} & \mathbf{1} \quad \mathbf{2} \quad \mathbf{3} & \\ [m] = & \begin{bmatrix} 3.2 & 1.6 & 0 \\ 1.6 & 6.4 & 1.6 \\ 0 & 1.6 & 3.2 \end{bmatrix} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \end{array} \qquad \dots (3)$$

∴ Global matrix, for longitudinal bar element,

$$\{[K] - \omega^2 [m]\}\{u\} = 0$$

$$\left\{10^5 \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 3.2 & 1.6 & 0 \\ 1.6 & 6.4 & 1.6 \\ 0 & 1.6 & 3.2 \end{bmatrix}\right\} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

Applying boundary conditions,

$u_1 = 0\ (fixed),$

$u_2 = u_2$

$u_3 = u_3$

$$\left\{10^5 \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 3.2 & 1.6 & 0 \\ 1.6 & 6.4 & 1.6 \\ 0 & 1.6 & 3.2 \end{bmatrix}\right\} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\left\{10^5\begin{bmatrix}12 & -6\\ -6 & 6\end{bmatrix} - \omega^2\begin{bmatrix}6.4 & 1.6\\ 1.6 & 3.2\end{bmatrix}\right\}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = 0 \qquad \text{... (4)}$$

To obtain a solution to the set of homogeneous equation in equation (4), we set the determinant of the coefficient matrix equal to zero.

$$\left|10^5\begin{bmatrix}12 & -6\\ -6 & 6\end{bmatrix} - \omega^2\begin{bmatrix}6.4 & 1.6\\ 1.6 & 3.2\end{bmatrix}\right| = 0 \qquad \text{... (5)}$$

$$\Rightarrow \quad \begin{vmatrix}(12\times 10^5 - 6.4\omega^2) & (-6\times 10^5 - 1.6\omega^2)\\ (-6\times 10^5 - 1.6\,\omega^2) & (6\times 10^5 - 3.2\,\omega^2)\end{vmatrix} = 0$$

$$\Rightarrow \quad [(12\times 10^5 - 6.4\omega^2)(6\times 10^5 - 3.2\,\omega^2)] - [(-6\times 10^5 - 1.6\,\omega^2)(-6\times 10^5 - 1.6\,\omega^2)] = 0$$

$$\Rightarrow \quad [(7.2\times 10^{11} - 3.84\times 10^6\,\omega^2 - 3.84\times 10^6\omega^2 + 20.48\,\omega^4)] - [(3.6\times 10^{11} + 9.6\times 10^6\,\omega^2 + 9.6\times 10^5\omega^2 + 2.56\,\omega^2)] = 0$$

$$\Rightarrow \quad 3.6\times 10^{11} - 9.6\times 10^6\omega^2 + 17.92\,\omega^4 = 0$$

$Let,$ $\quad \lambda = \omega^2$

$$17.92\,\lambda^2 - 9.60\times 10^6 + 3.6\times 10^{11} = 0$$

$$\lambda^2 - 53.57\times 10^4\,\lambda + 2\times 10^{10} = 0 \qquad \text{... (6)}$$

By solving the quadratic equation (6),

$$\Rightarrow \quad \lambda = \frac{53.57\times 10^4 \pm \sqrt{(53.57\times 10^4)^2 - 4(2\times 10^{10})}}{2(1)}$$

$$\left[\because \lambda = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}\right]$$

$$\Rightarrow \quad \lambda_1 = -40218.683$$

$$\Rightarrow \quad \lambda_2 = -497281.316$$

We know that, natural frequencies,

$$\omega^2 = \lambda$$

$$\Rightarrow \quad \omega = \sqrt{\lambda}$$

$\therefore \qquad \omega_1 = \sqrt{\lambda_1} = \sqrt{40218.683}$

$$\omega_1 = 200.54\, rad/s$$

Similarly, $\qquad \omega_2 = \sqrt{\lambda_2} = \sqrt{497281.316}$

$$\omega_2 = 705.181\, rad/s$$

Result: Natural frequencies of longitudinal are,

$$\omega_1 = 200.54\, rad/s$$

$$\omega_2 = 705.181\, rad/s$$